TYPE D SPACES

AND

QUASI-DIAGONALIZABILITY

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CONTENTS

ABSTRACT

The coordinate system of Kerr and Debney is used to find the empty Type D metrics with a diverging principal null vector. These spaces are shown to be precisely that subclass of the diverging, empty, algebraically special spaces which are quasi-diagonalizable. This leads to the canonical forms found by Plebanski and Demianski for the empty Type D metrics. These are generalized to a class of charged Type D metrics possessing a cosmological constant.

The theory of symmetries in an empty algebraically special space is examined, revealing that those spaces with two commuting Killing vectors are characterized by four real constants, and that if two of these are zero, the space is Type D, and quasi-diagonalizable. The field equations are then linearized, and solved completely.

A brief discussion of conformal Killing tensors is given, and an upper bound is found for the number of linearly independent, second order, trace-free, conformal Killing tensors in any Riemannian space of dimension greater than two. Finally, it is shown that the Type D metrics are a natural subclass of those spaces with quasi-diagonal metrics and non-redundant, conformal Killing tensors.

CHAPTER I

INTRODUCTION

In 1969 William Kinnersley[1] found that empty Type D metrics have two commuting Killing vectors. About this time Kerr and Debney[2] discussed the problem of finding empty algebraically special spaces[3] with two commuting Killing vectors, but they did not discuss the Type D metrics explicitly.

In this thesis the Type D metrics are solved in the coordinate system of Kerr and Debney, allowing a direct interplay between their results and those of Kinnersley. Consequently, progress has been made on the theory of algebraically special spaces with two commuting Killing vectors. It is shown that these spaces are characterized by four real constants, and that the space is Type D if a particular pair of these is zero. The field equations[4] may then be linearized, and solved completely.

The theory of Type D spaces may also be extended. We shall concentrate on those empty Type D metrics with diverging principal null vectors[5] and show that they are particularly simple spaces - the most general member being described by a quartic function of a single variable. This simplicity is traced to the quasi-diagonalizability (q.d. for short) of the empty Type D spaces.

We shall understand this property to mean that the space allows two commuting Killing vectors and that two non-ignorable coordinates can be chosen in such a way that there are no cross-terms in the metric between them and the ignorable Killing coordinates. In Chapter V and VI we prove that an empty algebraically special space with a diverging principal null vector is Type D iff it is q.d..

The importance of q.d. was recognized soon after the discovery of the Kerr metric.[6] In 1966 A. Papapetrou[7] showed that an empty metric with two commuting Killing vectors has an axis only if it is q.d.. This result was generalized by Carter who established the relationship between q.d. (which he called orthogonal transitivity) and the existence of horizons.[8]

There are two sections to this dissertation. In Section A we discuss the relationship between empty algebraically special spaces with two commuting Killing vectors and the Type D spaces. Maxwell fields and conformal Killing tensors[9] (C.K.T.'s for short) are introduced in Section B to generalize these empty Type D spaces.

The motivation for discussing C.K.T.'s stems from a paper by Walker and Penrose,[10] in which it was shown that the empty Type D spaces have one. This result generalizes some work by Carter,[11] who found that the non-radiating Type D metrics have a Killing tensor. In fact, the existence of a non-redundant[12] C.K.T. in an empty Type D space guarantees a conformally separable coordinate system[13] for the Hamilton-Jacobi equation, and it is this coordinate system which quasi-diagonalizes the metric.

The notation used in later chapters is introduced in Chapter II. Chapter III contains a derivation of the conditions which an empty algebraically special space and a Type D space must satisfy. These are expressed in the coordinate system used by Kerr and Debney, thereby allowing us to draw directly from their results on the symmetries in an algebraically special empty space. In Chapter IV a proof of Kinnersley's theorem that the empty Type D spaces have two commuting Killing vectors is given. In addition, definitions of radiating and non-radiating spaces[14] are given, together with a derivation of the non-radiating Type D spaces.

Chapter V contains the solution of the radiating Type D metrics, and also Papapetrou's concept of quasi-diagonalizability. This leads to the canonical form for the Kinnersley metric, found by J. Plebanski and M. Demianski.[15] The results of Chapter V are extended in Chapter VI to the non-radiating Type D spaces. A Maxwell field and a cosmological constant are introduced in Chapter VII, and generalizations of the Type C and Kinnersley metrics found. These are analogous to the generalization of the empty Kerr-NUT space found by Carter. Chapter VIII contains a discussion of conformal Killing tensors, which are used in Chapter IX to find generalizations of the Boyer-Lindquist[16] metrical form for the Kerr-NUT space.

CHAPTER II

SECTION A NOTATION

We shall suppose that $\{e_{A\dot{B}}\}$ (†) and $\{\partial_\alpha\}$ (††) are spin (hermitean) and holonomic bases respectively for vector fields on an Einstein space, $\mathcal{E}$. Any vector, V, may be expressed in component form with respect to either base,

$$V = V^{A\dot{B}} e_{A\dot{B}} = V^\alpha \partial_\alpha . \qquad (2.1)$$

The dual spin basis, $\{\omega^{A\dot{B}}\}$, is defined in terms of the holonomic dual basis, $\{dx^\alpha\}$, through the equations

$$\omega^{A\dot{B}} = \omega^{A\dot{B}}{}_\alpha \, dx^\alpha , \quad \omega^{A\dot{B}}(e_{C\dot{D}}) = \delta^A_C \, \delta^{\dot{B}}_{\dot{C}} . \qquad (2.2)$$

The complex conjugate of a spinor (†††) is

$$T^{\dot{A}\ldots\ B\ldots}_{C\ldots\ \dot{D}\ldots} = \overline{T^{A\ldots\ \dot{B}\ldots}_{\dot{C}\ldots\ D\ldots}} . \qquad (2.3)$$

In future we shall adopt the notation that a vector, V, be written as $V = V^\alpha \partial_\alpha = V^{A\dot{A}} e_{A\dot{A}}$. The advantage that the indices A and $\dot{A}$ are naturally associated with the index α more than compensates for the additional care which must be taken in treating A and $\dot{A}$ as independent, and when performing complex conjugation.

The set $\{e_{A\dot{A}}\}$ is chosen to be a set of null vectors, whose inner products are defined through the following equations

$$g_{A\dot{A}B\dot{B}} = g(e_{A\dot{A}}, e_{B\dot{B}}) = \epsilon_{AB} \, \epsilon_{\dot{A}\dot{B}} , \qquad (2.4)$$

(†) A,B,... ranges over 0,1 ; $\dot{A},\dot{B}$ ranges over $\dot{0},\dot{1}$.

(††) $\alpha,\beta,\ldots$ ranges over (1,2,3,4).

(†††) We shall understand a spinor to be a complex geometric object subject to certain formal rules of manipulation, and whose generic component form is $T^{A\ldots\ \dot{B}\ldots}_{\dot{C}\ldots\ldots\ldots}$.

$$\epsilon_{AB} = \epsilon_{[AB]} , \quad \epsilon_{01} = 1 . \tag{2.5}$$

Since the members of $\{e_{A\dot{A}}\}$ are hermitean, $e_{0\dot{0}}$ and $e_{1\dot{1}}$ are real null vectors, and $e_{0\dot{1}}$ and $e_{1\dot{0}}$ are complex null vectors. The metric, ds^2, is defined as

$$ds^2 = g_{\alpha\beta}\, dx^{\alpha} dx^{\beta} = g_{A\dot{A}B\dot{B}}\, \omega^{A\dot{A}}\, \omega^{B\dot{B}} , \tag{2.6a}$$

$$= 2(\omega^{0\dot{0}}\, \omega^{1\dot{1}} - \omega^{0\dot{1}}\, \omega^{1\dot{0}}) \quad . \tag{2.6b}$$

It is consistent to raise spinor indices with ϵ^{AB}, and lower them with ϵ_{AB}, where

$$\epsilon^{AB} = \epsilon^{[AB]} , \quad \epsilon^{01} = 1 \quad . \tag{2.7a}$$

$$\zeta^{A} = \epsilon^{AB}\, \zeta_{B} , \quad \zeta_{A} = - \epsilon_{AB}\, \zeta^{B} , \tag{2.7b}$$

$$\epsilon_{A}{}^{B} = \delta_{A}^{B} \quad . \tag{2.7c}$$

A covariant derivative, ∇, which acts on spinors and agrees with the standard covariant derivative for tensors, is defined through the equations

$$\nabla T^{\dot{A}B..}{}_{C..} = dT^{\dot{A}B..}{}_{C..} + \Gamma^{\dot{A}}{}_{\dot{X}}\, T^{\dot{X}B..}{}_{C..} + \Gamma^{B}{}_{X}\, T^{\dot{A}X..}{}_{C..} - \Gamma^{X}{}_{C}\, T^{\dot{A}B..}{}_{X..} , \tag{2.8}$$

$$\Gamma_{AB} = \Gamma_{(AB)} \quad . \tag{2.9}$$

The $\Gamma^{A}{}_{B}$ are called the spin connection one-forms.

The Riemann tensor is defined through the equation $\nabla_{[\gamma}\, \nabla_{\delta]}\, T_{\beta} = \frac{1}{2} R^{\alpha}{}_{\beta\gamma\delta}\, T_{\alpha}$, giving the following spinor components for the Riemann tensor[17],

$$R_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = \psi_{ABCD}\, \in_{\dot{A}\dot{B}}\, \in_{\dot{C}\dot{D}} + \psi_{\dot{A}\dot{B}\dot{C}\dot{D}}\, \in_{AB}\, \in_{CD}$$

$$+ \in_{AB}\, \in_{\dot{C}\dot{D}}\, \Phi_{CD\dot{A}\dot{B}} + \in_{CD}\, \in_{\dot{A}\dot{B}}\, \Phi_{AB\dot{C}\dot{D}}$$

$$+ 2\Lambda(\in_{AC}\, \in_{BD}\, \in_{\dot{A}\dot{B}}\, \in_{\dot{C}\dot{D}} + \in_{AB}\, \in_{CD}\, \in_{\dot{A}\dot{D}}\, \in_{\dot{B}\dot{C}}) \,, \qquad (2.10)$$

$$\psi_{ABCD} = \psi_{(ABCD)} \quad , \qquad (2.11a)$$

$$\Phi_{AB\dot{C}\dot{D}} = \Phi_{(AB)(\dot{C}\dot{D})} = \overline{\Phi_{AB\dot{C}\dot{D}}} \quad , \qquad (2.11b)$$

$$\Lambda = \bar{\Lambda} \quad , \qquad (2.11c)$$

$$24\Lambda = R^{\alpha\beta}{}_{\alpha\beta} \quad . \qquad (2.11d)$$

$\mathcal{S}$ is conformally flat if $\psi_{ABCD} = 0$; it is empty if $\Phi_{AB\dot{C}\dot{D}} = 0 = \Lambda$. The components of the Weyl spinor, ψ_{ABCD}, are defined through the equations

$$\psi_i = \psi_{ABCD} \,, \quad i = A + B + C + D \,. \qquad (2.12)$$

The Cartan structural equations in spinor form are

$$d\omega_{A\dot{A}} = (\in_{\dot{A}\dot{B}}\, \Gamma_{ABC\dot{C}} + \in_{AB}\, \Gamma_{\dot{A}\dot{B}C\dot{C}})\, \omega^{C\dot{C}} \wedge \omega^{B\dot{B}} \,, \qquad (2.13a)$$

$$d\Gamma_{AB} + \Gamma_{AC} \wedge \Gamma^{C}{}_{B} = \tfrac{1}{2}\psi_{ABCD}\, \omega^{C}{}_{\dot{D}} \wedge \omega^{D\dot{D}}$$

$$+ \tfrac{1}{2}\Phi_{AB\dot{C}\dot{D}}\, \omega_{D}{}^{\dot{C}} \wedge \omega^{D\dot{D}} + \Lambda\, \omega_{A\dot{D}} \wedge \omega_{B}{}^{\dot{D}} \,, \qquad (2.13b)$$

$$\Gamma_{AB} = \Gamma_{ABC\dot{C}}\, \omega^{C\dot{C}} \,. \qquad (2.13c)$$

It is sometimes useful to use the Newman - Penrose notation[18] for the spin coefficients.

$$\begin{aligned}
&\Gamma_{000\dot{0}} = \kappa \,, && \Gamma_{010\dot{0}} = \varepsilon \,, && \Gamma_{110\dot{0}} = \pi \,, \\
&\Gamma_{001\dot{0}} = \rho \,, && \Gamma_{011\dot{0}} = \alpha \,, && \Gamma_{111\dot{0}} = \lambda \,, \\
&\Gamma_{000\dot{1}} = \sigma \,, && \Gamma_{010\dot{1}} = \beta \,, && \Gamma_{110\dot{1}} = \mu \,, \\
&\Gamma_{001\dot{1}} = \tau \,, && \Gamma_{011\dot{1}} = \gamma \,, && \Gamma_{111\dot{1}} = \nu \,.
\end{aligned} \qquad (2.14)$$

The last two indices, $C\dot{C}$, in $\Gamma_{ABC\dot{C}}$ are hermitean, and so, for example, $\Gamma_{\dot{0}1\dot{0}\dot{1}} = \bar{\alpha}$. With this notation, eq.(2.13a) may be written as

$$d\omega_{0\dot{0}} = \kappa\omega^{0\dot{0}} \wedge \omega^{0\dot{1}} + \bar{\kappa}\omega^{0\dot{0}} \wedge \omega^{1\dot{0}} + (\varepsilon + \bar{\varepsilon})\, \omega^{0\dot{0}} \wedge \omega^{1\dot{1}}$$

$$+ (\bar{\rho} - \rho)\, \omega^{0\dot{1}} \wedge \omega^{1\dot{0}} + (\bar{\alpha} + \beta - \tau)\, \omega^{0\dot{1}} \wedge \omega^{1\dot{1}}$$

$$+ (\alpha + \bar{\beta} - \bar{\tau})\, \omega^{1\dot{0}} \wedge \omega^{1\dot{1}}, \tag{2.15a}$$

$$d\omega_{0\dot{1}} = \sigma\omega^{0\dot{0}} \wedge \omega^{1\dot{0}} + (\rho - \varepsilon + \bar{\varepsilon})\, \omega^{0\dot{0}} \wedge \omega^{1\dot{0}} + (\tau + \bar{\pi})\, \omega^{0\dot{0}} \wedge \omega^{1\dot{1}}$$

$$+ (\bar{\alpha} - \beta)\, \omega^{0\dot{1}} \wedge \omega^{1\dot{0}} + \bar{\lambda}\omega^{0\dot{1}} \wedge \omega^{1\dot{1}} + (\gamma - \bar{\gamma} + \bar{\mu})\, \omega^{1\dot{0}} \wedge \omega^{1\dot{1}}, \tag{2.15b}$$

$$d\omega_{1\dot{1}} = (\bar{\alpha} + \beta - \bar{\pi})\, \omega^{0\dot{0}} \wedge \omega^{0\dot{1}} + (\alpha + \bar{\beta} - \pi)\, \omega^{0\dot{0}} \wedge \omega^{1\dot{0}}$$

$$+ (\gamma + \bar{\gamma})\, \omega^{0\dot{0}} \wedge \omega^{1\dot{1}} + (\bar{\mu} - \mu)\, \omega^{0\dot{1}} \wedge \omega^{1\dot{0}}$$

$$+ \bar{\nu}\omega^{0\dot{1}} \wedge \omega^{1\dot{1}} + \nu\omega^{1\dot{0}} \wedge \omega^{1\dot{1}}. \tag{2.15c}$$

Eqs.(2.13) are the Newman - Penrose (hereafter referred to as the [N.P.]) equations.

The source-free Maxwell equations are

$$\nabla^{A\dot{A}}\, \phi_{AB} = 0 \quad , \tag{2.16a}$$

$$\phi_{AB} = \phi_{(AB)} \quad , \tag{2.16b}$$

$$\Phi_{AB\dot{C}\dot{D}} = \phi_{AB}\, \phi_{\dot{C}\dot{D}}\, . \tag{2.16c}$$

Since ϕ_{AB} is symmetric, it is convenient to write

$$\phi_i = \phi_{AB}\, , \qquad i = A + B\, . \tag{2.17}$$

The spin transformations are the linear maps, $\zeta^{A'} = S^{A'}{}_{A}\,\zeta^{A}$, for which $\in_{AB}$ is invariant. Then,

$$T^{\dot{A}'B'}{}_{C'} = S^{\dot{A}'}{}_{\dot{A}}\, S^{B'}{}_{B}\, S^{C}{}_{C'}\, T^{\dot{A}B}{}_{C} \,. \tag{2.18}$$

From the definition of $S^{A'}{}_{A'}$, its inverse, $S^{A}{}_{A'}$, and determinant are given by

$$S^{C}{}_{C'} = \in^{CD} \in_{C'D'} S^{D'}{}_{D}\,, \tag{2.19}$$

$$\det \left(S^{A'}{}_{A}\right) = \tfrac{1}{2} \in^{AB} \in_{A'B'} S^{A'}{}_{A}\, S^{B'}{}_{B} = 1\,. \tag{2.20}$$

The conformal tensor, and the connection transform as follows,

$$\psi_{A'B'C'D'} = S^{A}{}_{A'}\, S^{B}{}_{B'}\, S^{C}{}_{C'}\, S^{D}{}_{D'}\, \psi_{ABCD}\,, \tag{2.21}$$

$$\Gamma^{A'}{}_{B'} = S^{A'}{}_{A}\, S^{B}{}_{B'}\, \Gamma^{A}{}_{B} - S^{A}{}_{B'}\, dS^{A'}{}_{A}\,. \tag{2.22}$$

Finally, the Bianchi Identities are

$$\nabla^{D}{}_{\dot{Y}}\, \psi_{ABCD} = \nabla_{(C}{}^{\dot{Z}}\, \Phi_{AB)\dot{Y}\dot{Z}}\,, \tag{2.23a}$$

$$\nabla^{B\dot{Z}}\, \Phi_{AB\dot{Y}\dot{Z}} = -\,3\,\nabla_{A\dot{Y}}\, \Lambda\,. \tag{2.23b}$$

CHAPTER III

EMPTY ALGEBRAICALLY SPECIAL SPACES

The notation of Chapter II is used in this chapter to summarize those parts of the established theory of empty algebraically special spaces which we shall need later.

Let $\mathcal{S}$ be such a space, so that

$$\Phi_{AB\dot{C}\dot{D}} = 0 = \Lambda , \tag{3.1}$$

and suppose that $\omega^{1\dot{1}}$ is a fixed principal null vector (hereafter referred to as a p.n.v.) of $\mathcal{S}$. Under a spin transformation, $\omega^{1\dot{1}}$ must go into a multiple of itself, and so the remaining spin transformations can be parametrized by two complex functions, g and f,

$$S^{A'}{}_{A} = \begin{pmatrix} S^{0'}{}_{0} & S^{0'}{}_{1} \\ S^{1'}{}_{0} & S^{1'}{}_{1} \end{pmatrix} = \begin{pmatrix} f^{-1} & gf^{-1} \\ 0 & f \end{pmatrix} . \tag{3.2}$$

The parameter g gives a null rotation about $\omega^{1\dot{1}}$. From eq. (2.18) and eq. (2.22),

$$\omega^{1'\dot{1}'} = f\bar{f}\omega^{1\dot{1}} \quad , \tag{3.3a}$$

$$\omega^{0'\dot{1}'} = (\bar{f}/f)(\omega^{0\dot{1}} + g\omega^{1\dot{1}}) \quad , \tag{3.3b}$$

$$\omega^{0'\dot{0}'} = (f\bar{f})^{-1} (\omega^{0\dot{0}} + g\omega^{1\dot{0}} + \bar{g}\omega^{0\dot{1}} + g\bar{g}\omega^{1\dot{1}}) \; , \tag{3.3c}$$

$$\Gamma_{0'0'} = f^2 \Gamma_{00} \quad , \tag{3.4a}$$

$$\Gamma_{0'1'} = \Gamma_{01} + g\,\Gamma_{00} + d(\log f) \quad , \tag{3.4b}$$

$$\Gamma_{1'1'} = f^{-2} (\Gamma_{11} - 2g\,\Gamma_{01} + g^2\,\Gamma_{00} - dg) \quad . \tag{3.4c}$$

The first two conformal tensor components, ψ_0 and ψ_1, are both zero, since $\omega^{1\dot{1}}$ is a p.n.v., whilst from eq. (2.21), the others transform as

$$\psi_{2'} = \psi_2 \quad , \tag{3.5a}$$

$$\psi_{3'} = (\psi_3 - 3g\,\psi_2)f^{-2} \quad , \tag{3.5b}$$

$$\psi_{4'} = (\psi_4 - 4g\psi_3 + 6g^2\psi_2)f^{-4} \quad . \tag{3.5c}$$

For completeness, we shall outline a derivation of the field equations, the spin coefficients, $\Gamma_{ABC\dot{C}}$, and the Weyl components, ψ_i, for these spaces, as well as calculating the identity component of the group, $\mathcal{G}$, which consists of spin and coordinate transformations preserving the coordinate conditions imposed on $\mathcal{S}$. We shall follow closely the work of Debney, Kerr and Schild.[19]

The components of Γ_{00} are the optical scalars of Sachs[20] for the null vector $\omega^{1\dot{1}}$:

$\kappa = \Gamma_{000\dot{0}}$ = geodesy ,

$\rho = \Gamma_{001\dot{0}}$ = complex divergence ,

$\sigma = \Gamma_{000\dot{1}}$ = shear ,

$\tau = \Gamma_{001\dot{1}}$.

The Goldberg - Sachs theorem states that in an empty algebraically special space, two of these scalars, κ and σ, are zero. In $\mathcal{S}$, therefore,

$$\Gamma_{00} = \rho\omega^{1\dot{0}} + \tau\omega^{1\dot{1}} . \tag{3.6}$$

For the next five chapters we shall assume that ρ is non-zero.

Assumption : $\rho \neq 0$.

We shall now outline the solution of eq. (2.13). This involves the following sequence of operations : satisfy the (00) component of eq. (2.13b), the $(0\dot{1})$ component of eq. (2.13a), the $(0\dot{0})$ component of eq. (2.13a), the (01) component of eq. (2.13b), the $(1\dot{1})$ component of eq. (2.13a), and finally the (11) component of eq. (2.13b). We do not use the [N.P.] equations explicitly, since the structure of eq. (2.13) will be used to find some of our coordinates. Of course the [N.P.] equations will give any relationship which follows from eq. (2.13), but in general this may involve a technically difficult construction.

The (0 0) component of eq. (2.13b) reduces to

$$d\Gamma_{00} + 2\Gamma_{00} \wedge \Gamma_{01} = \psi_2\, \omega^{1\dot{0}} \wedge \omega^{1\dot{1}} , \tag{3.7}$$

and so, from eq. (3.6) ,

$$\Gamma_{00} \wedge d\Gamma_{00} = 0 . \tag{3.8}$$

The complete solution of this is $\Gamma_{00} = \eta d\zeta$, where η and ζ are both complex functions. A spin transformation (with $f^2 = -\eta$ and $g = \tau/\rho$) transforms Γ_{00} to $- d\zeta$, and τ to zero, so that

$$\Gamma_{00} = \rho\omega^{1\dot{0}} = - d\zeta . \tag{3.9}$$

Any transformation preserving eq. (3.9) must satisfy $d\zeta' = f^2 d\zeta$, and so

$$\zeta' = \Phi(\zeta) , \quad f^2 = \Phi_\zeta , \quad g = 0 , \tag{3.10}$$

where Φ is an analytic function, and f and g are defined in eq. (3.2). Substituting eq. (3.9) into eq. (3.7) gives

$$2\rho\omega^{1\dot{0}} \wedge \{\varepsilon\,\omega^{0\dot{0}} + \beta\omega^{0\dot{1}} + \alpha\omega^{1\dot{0}} + \gamma\omega^{1\dot{1}}\} = \psi_2\, \omega^{1\dot{0}} \wedge\, \omega^{1\dot{1}}\,, \qquad (3.11)$$

and thus

$$\psi_2 = 2\rho\gamma\ , \qquad (3.12a)$$

$$\varepsilon = \beta = 0\,. \qquad (3.12b)$$

If we substitute $\omega_{0\dot{1}} = \rho^{-1}\, d\zeta$ into eq.(2.13a), and use eq.(2.15b), we find

$$\begin{aligned} d\omega_{0\dot{1}} &= -\rho(\rho^{-1})_{0\dot{0}}\ \ \omega^{0\dot{0}} \wedge\, \omega^{1\dot{0}} - \rho(\rho^{-1})_{0\dot{1}}\ \ \omega^{0\dot{1}} \wedge\, \omega^{1\dot{0}} \\ &\qquad + \rho(\rho^{-1})_{1\dot{1}}\ \ \omega^{1\dot{0}} \wedge\, \omega^{1\dot{1}}\,, \\ &= (\rho - \varepsilon + \overline{\varepsilon})\, \omega^{0\dot{0}} \wedge\, \omega^{1\dot{0}} + \overline{\pi}\,\omega^{0\dot{0}} \wedge\, \omega^{1\dot{1}} + (\overline{\alpha} - \beta)\, \omega^{0\dot{1}} \wedge\, \omega^{1\dot{0}} \\ &\qquad + \overline{\lambda}\,\omega^{0\dot{1}} \wedge\, \omega^{1\dot{1}} + (\gamma - \overline{\gamma} + \overline{\mu})\, \omega^{1\dot{0}} \wedge\, \omega^{1\dot{1}}\,. \end{aligned}$$

Therefore

$$\rho_{0\dot{0}} = \rho^2\ , \qquad (3.13a)$$

$$\rho_{0\dot{1}} = \overline{\alpha}\rho\ , \qquad (3.13b)$$

$$\rho_{1\dot{1}} = -(\gamma - \overline{\gamma} + \overline{\mu})\,\rho\ , \qquad (3.13c)$$

$$\pi = 0 = \lambda\ . \qquad (3.13d)$$

Following Kerr and Debney, (hereafter referred to as [K.D.]), ζ and $\overline{\zeta}$ are used as coordinates, and a third real coordinate, u, is introduced through the integrable equations

$$u_{0\dot{0}} = 0\,,\quad u_{1\dot{1}} = 1\,, \qquad (3.14)$$

giving

$$\omega_{0\dot{0}} = du + \Omega d\zeta + \overline{\Omega} d\overline{\zeta}\,, \quad \Omega = \rho^{-1} u_{1\dot{0}} \,. \tag{3.15}$$

From eq.(3.3a) and eq.(3.10), any transformation preserving eq.(3.15) must satisfy

$$\omega^{1'\dot{1}'} = du' + \Omega' d\zeta' + \overline{\Omega}' d\overline{\zeta}' = |\Phi_\zeta| (du + \Omega d\zeta + \overline{\Omega} d\overline{\zeta}) \,, \tag{3.16}$$

and consequently

$$u' = |\Phi_\zeta| (u + U) \,, \tag{3.17}$$

$$\Omega' = (|\Phi_\zeta|/\Phi_\zeta) \, [\Omega - U_\zeta - (|\Phi_\zeta|_\zeta / |\Phi_\zeta|)(u + U)] \,, \tag{3.18}$$

where U is a real function of ζ and $\overline{\zeta}$. The fourth real coordinate, v, is chosen to be an affine parameter along $\omega^{1\dot{1}}$,

$$v_{0\dot{0}} = 1 \,. \tag{3.19}$$

From eq.(3.13a), v can be chosen as

$$v = -\operatorname{Re}(\rho^{-1}) \,. \tag{3.20}$$

Under a spin transformation, $\rho' = |f|^2 \rho$, and as $|f|^2 = |\Phi_\zeta|$, the allowed transformations for v and ρ are

$$v' = |\Phi_\zeta|^{-1} v \,, \tag{3.21a}$$

$$\rho' = |\Phi_\zeta| \rho \,. \tag{3.21b}$$

From eq.(3.9), eq.(3.14) and eq.(3.19), the operator $e_{0\dot{0}}$ is given by $e_{0\dot{0}} = \partial_v$. If eq.(3.15) is substituted into eq.(2.13a) and eq.(2.15a) is used, we find

$$\Omega_{0\dot{0}} = 0 \,, \tag{3.22a}$$

$$\overline{\rho\Omega}_{1\dot{0}} - \rho\Omega_{0\dot{1}} = \overline{\rho} - \rho \,, \tag{3.22b}$$

$$\alpha = \rho\Omega_{1\dot{1}} \quad , \tag{3.22c}$$

and so Ω is independent of v.

The next step is the (0 1) component of eq.(2.13b). Using $\Gamma_{00} = \rho\omega^{1\dot{0}}$, $\Gamma_{01} = \alpha\omega^{1\dot{0}} + \gamma\omega^{1\dot{1}}$, $\Gamma_{11} = \mu\omega^{0\dot{1}} + \nu\omega^{1\dot{1}}$ and the known results, we find that

$$(\alpha\rho^{-1})_{0\dot{0}} = 0 \quad , \tag{3.23a}$$

$$\psi_2 = \gamma_{0\dot{0}} \quad , \tag{3.23b}$$

$$\alpha_{0\dot{1}} = \rho\mu - \psi_2 + \gamma(\rho - \bar{\rho}) + \alpha\bar{\alpha} \quad , \tag{3.23c}$$

$$\gamma_{0\dot{1}} = -\bar{\alpha}\gamma \quad , \tag{3.23d}$$

$$\alpha_{1\dot{1}} - \gamma_{1\dot{0}} = \rho\nu + \alpha(\bar{\gamma} - \bar{\mu}) - \psi_3 \ . \tag{3.23e}$$

From eq.(3.12a), eq.(3.23b) and eq.(3.13a), we see that $(\gamma\rho^{-2})_{0\dot{0}} = 0$, and so

$$\gamma = -M\rho^2 \quad , \tag{3.24}$$

where M is a complex function of $u,\zeta,\bar{\zeta}$. We shall eventually show that the metric can be expressed as a function of Ω and M. The transformation properties of M follow from eq.(3.12a), eq.(3.21) and the invariance of ψ_2 .

$$M' = |\Phi_\zeta|^{-3} M \ . \tag{3.25}$$

We shall now consider $\omega_{1\dot{1}}$. From eq.(3.19),

$$\omega_{1\dot{1}} = dv + \iota d\zeta + \bar{\iota} d\bar{\zeta} + B\omega^{1\dot{1}} \quad , \tag{3.26a}$$

$$\iota = \rho^{-1} v_{1\dot{0}} \quad , \quad B = -v_{1\dot{1}} \quad . \tag{3.26b}$$

The following dual spin basis for one-forms on $\mathcal{B}$ may be deduced by using eq. (3.26), eq. (3.15) and eq. (3.9).

$$\omega^{0\dot{0}} = B\,du + dv + (B\Omega + \iota)\,d\zeta + (B\bar{\Omega} + \bar{\iota})\,d\bar{\zeta} \,, \tag{3.27a}$$

$$\omega^{1\dot{1}} = du + \Omega d\zeta + \bar{\Omega} d\bar{\zeta} \,, \tag{3.27b}$$

$$\omega^{0\dot{1}} = -\,\bar{\rho}^{-1}\,d\bar{\zeta} \,, \tag{3.27c}$$

$$\omega^{1\dot{0}} = -\,\rho^{-1}\,d\zeta \,, \tag{3.27d}$$

while, from eq. (2.2), we have the following spin basis for vector fields on $\mathcal{B}$,

$$e_{0\dot{0}} = \partial_v \,, \tag{3.28a}$$

$$e_{1\dot{1}} = \partial_u - B\partial_v \,, \tag{3.28b}$$

$$e_{0\dot{1}} = \overline{\rho\Omega}\partial_u + \overline{\rho\iota}\partial_v - \bar{\rho}\partial_{\bar{\zeta}} \,, \tag{3.28c}$$

$$e_{1\dot{0}} = \rho\Omega\partial_u + \rho\iota\partial_v - \rho\partial_\zeta \,. \tag{3.28d}$$

The remaining equations to be derived are found in exactly the same manner as shown above. In particular, the $(1\dot{1})$ component of eq. (2.13a) gives

$$\alpha = -\,\rho\iota_{0\dot{0}} \,, \tag{3.29a}$$

$$\gamma + \bar{\gamma} = B_{0\dot{0}} \,, \tag{3.29b}$$

$$\bar{\mu} - \mu = \bar{\rho}B\bar{\Omega}_{1\dot{0}} + \overline{\rho\iota}_{1\dot{0}} - \rho\iota_{0\dot{1}} - \rho B\Omega_{0\dot{1}} \,, \tag{3.29c}$$

$$\nu = \rho B\Omega_{1\dot{1}} + \rho\iota_{1\dot{1}} + B_{1\dot{0}} \,, \tag{3.29d}$$

while the (11) component of eq. (2.13b) gives

$$\psi_2 = \bar{\rho}[\mu(\bar{\rho})^{-1}]_{0\dot{0}} \quad , \tag{3.30a}$$

$$\psi_3 = \nu_{0\dot{0}} \quad , \tag{3.30b}$$

$$\psi_3 = 2\alpha\mu + \rho\nu - \bar{\rho}\nu + \bar{\rho}[\mu(\bar{\rho})^{-1}]_{1\dot{0}} \quad , \tag{3.30c}$$

$$2\mu\gamma = \nu_{0\dot{1}} + \bar{\alpha}\nu - \bar{\rho}[\mu(\bar{\rho})^{-1}]_{1\dot{1}} \quad , \tag{3.30d}$$

$$\psi_4 = 2\alpha\nu + \nu_{1\dot{0}} + \alpha\nu \quad . \tag{3.30e}$$

Since eq.(3.13) through to eq.(3.30) have been solved by Debney, Kerr and Schild, we shall only outline an explicit solution of them here. First, we shall introduce the definitions

$$D = \partial_\zeta - \Omega\partial_u = -\rho^{-1}e_{1\dot{0}} - \imath\partial_v \quad , \tag{3.31a}$$

$$H = \bar{D}\partial_u D\Omega \quad , \tag{3.31b}$$

$$I = DM - 3M\partial_u\Omega \quad , \tag{3.31c}$$

$$J = DD\bar{\Omega} - \bar{D}D\Omega \quad . \tag{3.31d}$$

The only non-zero spin connections are $\rho,\alpha,\gamma,\mu,\nu$; these are found from eq.(3.20), eq.(3.22c), eq.(3.24), eq.(3.23c) and eq.(3.29d) respectively.

$$\rho = -(v + \Delta)^{-1} \quad , \tag{3.32a}$$

$$\Delta = i\,\mathrm{Im}(D\bar{\Omega}) \quad , \tag{3.32b}$$

$$\alpha = \rho\partial_u\Omega \quad , \tag{3.32c}$$

$$\gamma = -M\rho^2 \quad , \tag{3.32d}$$

$$\mu = -\overline{\rho D}\partial_u\Omega - M(\rho^2 + \rho\bar{\rho}) \quad , \tag{3.32e}$$

$$\nu = \partial_u \partial_u \bar{\Omega} + \rho H + \rho^2 I + \rho^3 M \quad , \tag{3.32f}$$

$$\iota = - v \partial_u \Omega + \tfrac{1}{2}(\bar{D} D \Omega - D D \bar{\Omega}) \quad , \tag{3.32g}$$

$$B = - \mathrm{Re}[\, \bar{D} \partial_u \Omega + 2M\rho] \quad , \tag{3.32h}$$

where the relationships for Δ, ι, B follow from eq.(3.22b), eq.(3.22c) and eq.(3.13b), eq.(3.13c) respectively; and these have been used in an obvious manner to find some of the spin coefficients.

The Weyl components, ψ_2, ψ_3, ψ_4, are calculated from eq.(3.32) and eq.(3.12a), eq.(3.23e), eq.(3.30e) respectively.

$$\psi_2 = - 2M\rho^3 \quad , \tag{3.33a}$$

$$\psi_3 = H\rho^2 + 2I\rho^3 + 3MJ\rho^4 \quad , \tag{3.33b}$$

$$\begin{aligned} \psi_4 = {} & - (\partial_u \partial_u D\Omega)\rho - \rho^2 (D - 4\partial_u \Omega) H - \rho^3 [\, (D - 5\partial_u \Omega) I - HJ] \\ & - \rho^4 [\, (D - 6\partial_u \Omega)(MJ) + 2IJ] - 3MJ^2 \rho^5 \ . \end{aligned} \tag{3.33c}$$

The remaining equations in eq.(3.23), eq.(3.29) and eq.(3.30) involve only the variables listed in eq.(3.32) and eq.(3.33), and so they either are identities, or give relationships between M and Ω. There are three such non-trivial equations,

$$\text{F.E.I} \qquad \bar{D} M = 3M \partial_u \bar{\Omega} \quad , \tag{3.34a}$$

$$\text{F.E.II} \qquad \mathrm{Im}(2M - \overline{DDD}\Omega) = 0 \quad , \tag{3.34b}$$

$$\text{F.E.III} \qquad 2\partial_u M = (\bar{D} - 2\partial_u \bar{\Omega}) H \quad , \tag{3.34c}$$

coming from eq.(3.23c), eq.(3.29c) and eq.(3.30d) respectively. We shall call these field equations. The remaining equations are identities.

These field equations differ slightly from those in [K.D.], since we have replaced μ (there) by $2M$, and have rearranged F.E.III into a more convenient form for the purposes of integration. They are necessary and sufficient conditions for the tetrad defined in eq. (3.27) and eq. (3.32) to give an algebraically special, empty Einstein space.

This space is Type D iff there exists a g for which $\omega^{0'\dot{0}'}$ is also a p.n.v.; i.e. for which $\psi_{3'} = 0 = \psi_{4'}$. From eq. (3.5), $\mathcal{K}$ is Type D iff

$$\psi_0 = \psi_1 = 3\psi_2\,\psi_4 - 2\psi_3^2 = 0\,. \tag{3.35}$$

If we adopt the self explanatory notation that $\psi_2 = \overset{0}{\psi}_2\rho^3$, $\psi_3 = \overset{0}{\psi}_3\rho^2 + \overset{1}{\psi}_3\rho^3 + \overset{2}{\psi}_3\rho^4$, $\psi_4 = \overset{0}{\psi}_4\rho + \overset{1}{\psi}_4\rho^2 + \overset{2}{\psi}_4\rho^3 + \overset{3}{\psi}_4\rho^4$, then eq. (3.35) may be written as

$$2(\overset{0}{\psi}_3)^2 = 3\,\overset{0}{\psi}_2\overset{0}{\psi}_4\quad, \tag{3.36a}$$

$$4\overset{0}{\psi}_3\,\overset{1}{\psi}_3 = 3\,\overset{0}{\psi}_2\overset{1}{\psi}_4\quad, \tag{3.36b}$$

$$2(\overset{1}{\psi}_3)^2 + 4\,\overset{0}{\psi}_3\overset{2}{\psi}_3 = 3\,\overset{0}{\psi}_2\,\overset{2}{\psi}_4\quad, \tag{3.36c}$$

$$4\,\overset{1}{\psi}_3\,\overset{2}{\psi}_3 = 3\,\overset{0}{\psi}_2\,\overset{3}{\psi}_4\quad, \tag{3.36d}$$

$$2(\overset{2}{\psi}_3)^2 - 3\,\overset{0}{\psi}_2\overset{4}{\psi}_4 = \psi_0 = \psi_1 = 0\,. \tag{3.36e}$$

The last of these, eq. (3.36e), is satisfied in any algebraically special empty space. From eq. (3.33) and eq. (3.36), $\mathcal{K}$ is Type D iff

$$\text{D I}\qquad 3M\partial_u\partial_u D\Omega - H^2 = 0\;, \tag{3.37a}$$

$$\text{D II}\qquad D[H^3M^{-4}] = 0\;, \tag{3.37b}$$

$$\text{D III}\qquad 4(DM)^2 + 3MHJ - 3M(DDM) + 9M^2(\partial_u D\Omega) = 0\;, \tag{3.37c}$$

DIV $\quad D(J/M) = 0$. (3.37d)

The metric is flat iff $M = 0$, and so we must assume

$$M \neq 0 . \tag{3.38}$$

CHAPTER IV

ISOMETRIES OF $\mathcal{B}$

This chapter discusses the symmetries of Type D spaces. We begin by summarizing some results of [K.D.], who showed that for the metric of Chapter 3, any Killing vector, K, can be written as

$$K = \alpha\partial_\zeta + \bar{\alpha}\partial_{\bar{\zeta}} + \mathrm{Re}(\alpha_\zeta)(u\partial_u - v\partial_v) + P\partial_u , \tag{4.1}$$

where $\alpha^{(\dagger)}$ and P are certain functions of ζ, and $(\zeta,\bar{\zeta})$ respectively. The details leading to eq.(4.1) are given in Appendix (K.D.).

In Chapter 3 we saw that the metric is a function of M and Ω, and their derivatives. Since Lie differentiation and ordinary differentiation commute, the Killing equations reduce to $\mathcal{L}_K\Omega = 0 = \mathcal{L}_K M$, where $\mathcal{L}_K$ is the Lie derivative with respect to the K given in eq.(4.1). From the transformation properties of Ω and M in eq.(3.18) and eq.(3.25) respectively, we may write Killings' equations as

$$K\Omega + P_\zeta + \tfrac{1}{2}\alpha_{\zeta\zeta}\, u + \tfrac{1}{2}(\alpha_\zeta - \bar{\alpha}_{\bar{\zeta}})\,\Omega = 0 , \tag{4.2a}$$

$$KM + 3\mathrm{Re}(\alpha_\zeta)M = 0 . \tag{4.2b}$$

For completeness, eq.(4.2a) has been derived in Appendix (K.D.).

Under the coordinate group, $\mathcal{C}$, α transforms as $\alpha' = \Phi_\zeta\alpha$, and so if $\alpha \neq 0$, we may choose $\alpha = 1$; while if $\alpha = 0$, K has the simple form $K = P\partial_u$. The latter case features prominently in [K.D.].

(†) Here we use the notation of [K.D.]. α is not to be confused with a spin coefficient.

To facilitate further progress, we shall introduce the following definitions, which are motivated by the fact that all Type D metrics belong in a well-defined manner either to the set of metrics containing a Killing vector $P\partial_u$, or to its complement.

Definition 1: $\mathcal{D}$ is the set of all Type D spaces.

Definition 2: $\mathfrak{A}(n)$ is the set of algebraically special spaces for which ρ and M are non-zero, and which admit at least n commuting Killing vectors.

Definition 3: $\mathfrak{A}(1,N)$ is the set of metrics which admit a Killing vector of the type $P\partial_u$.

Definition 4: $\mathfrak{A}(1,R) = \mathfrak{A}(1) \setminus \mathfrak{A}(1,N)$.

Definition 5: $\mathfrak{A}(2,Q) = \mathfrak{A}(1,Q) \cap \mathfrak{A}(2)$, $Q = R,N$.

In the definitions above, R and N stand for radiating and non-radiating respectively, and we shall comment more fully on this at the end of this chapter. We shall not need to consider $\mathfrak{A}(n)$ for $n > 2$, since these were shown to be empty in [K.D.]. Furthermore, the set $\mathcal{D}$ has been shown by Kinnersley to be contained in $\mathfrak{A}(2)$. An independent proof of this will be given later.

If $\mathcal{S} \in \mathfrak{A}(1,N)$, it is convenient to introduce a slightly different coordinate system, $(\zeta,\bar{\zeta},r,s)$, defined by

$$s = u/P\ , \quad r = vP\ . \tag{4.3}$$

The metric is then independent of s, and can be written as

$$d\tau^2 = -2\Sigma P^{-2}\left|d\zeta - iP^2\,\delta_{\bar{\zeta}}\omega/\Sigma\right|^2 + (2dr + W\omega/\Sigma)\,\omega\ , \tag{4.4}$$

where

$$\omega = P^{-1}\,\omega^{1i} = ds + \Lambda d\zeta + \bar{\Lambda} d\bar{\zeta} \;, \qquad (4.5a)$$

$$\Lambda = (\Omega + sP_{\zeta})\,P^{-1} \;, \qquad (4.5b)$$

$$m = -\,2MP^{-3} \;, \qquad (4.5c)$$

$$\delta = -\,i\Delta P = P^{2}\,\mathrm{Im}(\bar{\Lambda}_{\zeta}) \;, \qquad (4.5d)$$

$$Q = \delta P \;, \qquad (4.5e)$$

$$\Sigma = P^{2}/\rho\bar{\rho} = r^{2} + \delta^{2} \;, \qquad (4.5f)$$

$$W = -\,2K_{2} - 2\mathrm{Re}(m)\,r + 2K_{1}r^{2} \;, \qquad (4.5g)$$

and K_1 and K_2 are the curvature invariants for the metrics $d\zeta d\bar{\zeta}/P^2$ and $d\zeta d\bar{\zeta}/Q^2$ respectively;

$$K_{1} = PP_{\zeta\bar{\zeta}} - P_{\zeta}P_{\bar{\zeta}} \;, \quad K_{2} = QQ_{\zeta\bar{\zeta}} - Q_{\zeta}Q_{\bar{\zeta}} \;. \qquad (4.6)$$

All these functions are independent of r and s, except for W and Σ, which are quadratic in r.

The field equations for this metric are found by substituting

$$\partial_{u} = P^{-1}\,\partial_{s} \;, \qquad (4.7a)$$

$$\partial_{v} = P\partial_{r} \;, \qquad (4.7b)$$

$$\partial_{\zeta} = -\,u\,P_{\zeta}P^{-2}\partial_{s} + vP_{\zeta}\partial_{r} + \partial_{\zeta} \;, \qquad (4.7c)$$

into eq. (3.34) and using a result, $\dot{\Omega} = -\,(\log P)_{\zeta}$, which follows from eq. (4.2). We find

$$\text{F.E.I} \qquad m_{\bar{\zeta}} = 0 \;, \qquad (4.8a)$$

$$\text{F.E.II} \qquad \mathrm{Im}(m) = PQ_{\zeta\bar{\zeta}} - P_{\zeta}Q_{\bar{\zeta}} - P_{\bar{\zeta}}Q_{\zeta} + P_{\zeta\bar{\zeta}}Q\,, \tag{4.8b}$$

$$\text{F.E.III} \qquad K_{1,\zeta\bar{\zeta}} = 0\,. \tag{4.8c}$$

The metrics in $\mathcal{D}(2,N)$ possess, in addition to the Killing vector ∂_s, a motion of the type[21]

$$K = \alpha\partial_{\zeta} + \bar{\alpha}\partial_{\bar{\zeta}} + a_0(s\partial_s - r\partial_r) + T\partial_s\,, \tag{4.9a}$$

$$[\partial_s, K] = a_0\partial_s\,. \tag{4.9b}$$

The vector K is a symmetry iff

$$KP + (a_0 - \mathrm{Re}(\alpha_{\zeta}))P = 0\,, \tag{4.10a}$$

$$Km + 3a_0 m = 0\,, \tag{4.10b}$$

$$K\delta + a_0\delta = 0\,, \tag{4.10c}$$

where α is an analytic function of ζ, and a_0 is a real constant. These equations do not depend on T, since P,m and δ are independent of s. Finally, T is any real function of ζ and $\bar{\zeta}$ satisfying the integrable equations,

$$T_{\zeta} + K\Lambda + (\alpha_{\zeta} - a_0)\,\Lambda = 0\,. \tag{4.11}$$

We shall simply state the result of [K.D.] for the $\mathcal{D}(2,R)$ metrics: the $(\zeta,\bar{\zeta},u,v)$ coordinates may be chosen with ∂_{ζ} as a complex Killing vector.

Having summarized that part of [K.D.] which we shall need, we begin our discussion of the Type D metrics by proving the following theorem of Kinnersley.

<u>Theorem 1:</u> Every Type D metric admits two commuting Killing vectors, and so $\mathcal{D}$ is the disjoint union of $\mathcal{D}(2,R)$ and $\mathcal{D}(2,N)$,

where $\mathcal{D}(2,Q) = \mathcal{D} \cap \mathcal{O}\!\mathcal{L}(2,R)$. $\quad Q = R,N.$

Proof: A proof of this will be given since rather different coordinates are being used to those of Kinnersley, and certain equations needed later will be derived.

If eq. (D.I.) is differentiated with respect to $\bar{D}$, and the first and third field equations used, then the following equation is obtained,

$$\partial_u [H^3 M^{-4}] = 0 \ . \tag{4.12}$$

Since eq. (D.II) is just

$$D[H^3 M^{-4}] = 0 \ , \tag{4.13}$$

we see that

$$H = \bar{f}(\bar{\zeta}) \ M^{4/3} \ , \tag{4.14}$$

where f is an analytic function of ζ. If the coordinate transformation in eq. (3.10) is applied,

$$(\partial_u D\Omega)' = \Phi_\zeta^{-2}[\partial_u D\Omega - \tfrac{1}{2}(\log \Phi_\zeta)_{\zeta\zeta} + \tfrac{1}{4}((\log \Phi_\zeta)_\zeta)^2] \ , \tag{4.15a}$$

$$H' = \bar{\Phi}_\zeta \ H/|\Phi_\zeta|^4 \ , \tag{4.15b}$$

we see that $f' = \Phi_\zeta f$. The function f can therefore be transformed to a constant, $-3\lambda_0$ (say), by an appropriate choice of Φ. We shall complete the proof of Theorem 1 by proving the following two lemmas.

Lemma 1R: The Type D metrics for which $H \neq 0$ are precisely those of Type $\mathcal{D}(2,R)$.

Proof: We may replace M by $\bar{N}$, where

$$\bar{N} = M^{-1/3} \ \lambda_0^{-1} \neq 0 \ , \tag{4.16}$$

since $H \neq 0$ implies $\lambda_o \neq 0$. Then, from eq.(4.14) and eq.(D.I), we find

$$\overline{D}\partial_u D\Omega = -3\lambda_0^{-3}\overline{N}^{-4} \quad , \tag{4.17a}$$

$$\partial_u\partial_u D\Omega = 3\lambda_0^{-3}\overline{N}^{-5} \quad . \tag{4.17b}$$

If eq.(4.17a) is substituted into eq.(F.E.III), and eq.(F.E.I) used, $\partial_u(N-\Omega) = 0$, and so N can be written as

$$N = \Omega + \in(\zeta,\overline{\zeta}) \quad . \tag{4.18}$$

We shall now show that $\in$ may be transformed to zero.

From eq.(3.17), the transformation $u \rightarrow u + U(\zeta,\overline{\zeta})$ maps $N \rightarrow N$, $\Omega \rightarrow \Omega - U_\zeta$, $\in \rightarrow \in + U_\zeta$, and so $\in$ can be eliminated whenever $\in_{\overline{\zeta}} = \overline{\in}_\zeta$. We shall prove

$$(\in_{\overline{\zeta}} - \overline{\in}_\zeta)(N^2)_{uuu} = 0 \quad , \tag{4.19}$$

and show that $(N^2)_{uuu}$ is zero only for flat space.

The following two (equivalent) equations are obtained by substituting eq.(4.18) into eq.(F.E.I),

$$(\partial_\zeta + \in\partial_u)N = 0 \quad , \tag{4.20}$$

$$D\Omega + \in_\zeta + N\partial_u N = 0 \quad . \tag{4.21}$$

If we differentiate eq.(4.21) with respect to u, we obtain

$$\partial_u D\Omega = -\tfrac{1}{2}N^2{}_{uu} \quad . \tag{4.22}$$

From eq.(4.17),

$$(\overline{D} + \overline{N}\partial_u)\,\partial_u D\Omega = 0 \quad , \tag{4.23}$$

and so

$$(\partial_{\bar{\zeta}} + \bar{\in}\partial_u)(N^2)_{uu} = 0 \; . \tag{4.24}$$

Finally, the commutators of the different operators in eq.(4.20) and eq.(4.24) are

$$[\partial_\zeta + \in\partial_u, \partial_{\bar{\zeta}} + \bar{\in}\partial_u] = (\bar{\in}_\zeta - \in_{\bar{\zeta}})\partial_u ; [\partial_\zeta + \in\partial_u, \partial_u] = 0 \; , \tag{4.25}$$

and therefore if eq.(4.22) is differentiated by $(\partial_\zeta + \in\partial_u)$, we obtain eq.(4.19). From eq.(4.17b) and eq.(4.22), $(N^2)_{uuu} \neq 0$, and so $\in$ can be transformed to zero. Doing this, eq.(4.20) and eq.(4.24) then imply that $(N^2)_{uu}$ is a function of u alone, and so, from eq.(4.22), $\partial_u D\Omega$ is a function of u. This, together with eq.(4.17b), shows that N itself is independent of ζ and $\bar{\zeta}$, and

$$2M = -\mu_0\bar{\Omega}^{-3}, \quad \Omega = \Omega(u) \; , \quad \mu_0 = 2\lambda_0^{-3} \; . \tag{4.26}$$

Since the metric is a function of M, Ω and v, ∂_ζ must be a complex Killing vector. [K.D.] have shown that such metrics do not possess a Killing vector of the type $K = P\partial_u$, and therefore $\mathcal{R} \in \mathcal{D}(2,R)$. This statement, together with Lemma 1N, proves Lemma 1R.

<u>Lemma 1N</u>: The Type D metrics for which H = 0 are precisely those of Type $\mathcal{D}$ (2,N).

<u>Proof</u>: When H = 0, eq.(F.E.III) and ∂_u[eq.(F.E.I)] reduce to

$$\partial_u M = 0 = \partial_u\partial_u\Omega \; , \tag{4.27}$$

and so eq.(D.I) and eq.(D.II) will be satisfied iff

$$\partial_u(\partial_u D\Omega) = 0 = \bar{D}(\partial_u D\Omega) \; , \tag{4.28}$$

i.e. $\partial_u D\Omega = g(\zeta)$, where g is an analytic function of ζ. It is not an invariant though, and, from eq.(4.27) and eq.(4.15a), it can be transformed to zero, provided we can find a $\Phi = \Phi(\zeta)$ satisfying

$$\partial_u D\Omega = (\dot{\Omega})_\zeta - (\dot{\Omega})^2 \quad , \tag{4.29a}$$

$$= (\log \Phi_\zeta)_{\zeta\zeta}/2 - [(\log \Phi_\zeta)_\zeta]^2/4 , \tag{4.29b}$$

$$= g(\zeta) \quad . \tag{4.29c}$$

If we define the function $q(\zeta)$ by

$$q(\zeta) = \dot{\Omega}(\zeta,\bar{\zeta})\big|_{\bar{\zeta} = \text{constant}} , \tag{4.30}$$

then it is certainly well-defined for some values of $\bar{\zeta}$, and the equation $(\log \Phi_\zeta)_\zeta = 2q(\zeta)$ may be solved for Φ; whence $\partial_u D\Omega$ can be transformed to zero.

$$\partial_u D\Omega = 0 . \tag{4.31}$$

From eq.(F.E.I),

$$\Omega_u = -(\log \bar{M}^{-1/3})_\zeta , \tag{4.32}$$

and substituting this into eq.(4.31),

$$(M^{-1/3})_{\bar{\zeta}\bar{\zeta}} = 0 , \tag{4.33}$$

we see that $M^{-1/3}$ is linear in $\bar{\zeta}$. Eq.(D.II) can now be simplified to

$$(M^{-1/3})_{\zeta\zeta} = 0 , \tag{4.34}$$

and so $M^{-1/3}$ is bilinear in ζ and $\bar{\zeta}$. We shall eventually prove that it is in fact a real bilinear function multiplied by a complex constant.

From eq.(F.E.II) and eq.(D.IV),

$$D\bar{D}\bar{D}D\Omega \;-\; DDD\bar{D}\bar{\Omega} = 2D(M-\bar{M}) \;, \tag{4.35a}$$

$$M\bar{D}D[\,(\bar{D}D\Omega \;-\; D\bar{D}\bar{\Omega})\,/\,M] \;=\; 0 \;. \tag{4.35b}$$

Subtracting one from the other, and using the commutator relation for D and $\bar{D}$,

$$[\,D,\bar{D}] \;=\; (\bar{D}\Omega \;-\; D\bar{\Omega})\,\partial_u \;=\; -\,2\Delta\partial_u \;, \tag{4.36}$$

the following equation is obtained after a rather lengthy calculation,

$$6\bar{M}\Omega_u \;=\; 2\,(\log M)_\zeta \;\;[\,\bar{M} + 2\Delta\,(D\partial_u\bar{\Omega})\,] \;-\; 6\Delta DD\partial_u\bar{\Omega} \;. \tag{4.37}$$

The coefficient of 2Δ is

$$2\,(\log M)_\zeta\,\bar{\Omega}_{u\zeta} \;\;-\;\; 3\Omega_{u\zeta\zeta}\;,$$

which is zero, as can be proved by substituting eq.(4.32) into the $\bar{\zeta}$ derivative of eq.(4.34). Consequently, eq.(4.37) reduces to

$$\Omega_u \;=\; -\,(\log M^{-1/3})_\zeta \quad , \tag{4.38}$$

which, with eq.(4.32), gives

$$(M\,/\,\bar{M})_\zeta \;\;=\; 0 \;. \tag{4.39}$$

This equation, and its complex conjugate, shows that $(M/\bar{M})$ is independent of ζ, $\bar{\zeta}$, u, and so we may write

$$M = -\tfrac{1}{2}\,\mu_0\,P^{-3} \quad , \tag{4.40}$$

where μ_0 is a complex constant, and P is a real bilinear function of ζ and $\bar{\zeta}$.

$$P_{\zeta\bar{\zeta}} \;=\; P_u \;\;=\; P \;-\; \bar{P} = 0 \;\;, \tag{4.41a}$$

$$\Omega_u \;\;=\; -\,(\log P)_\zeta \quad , \;\Delta_u \;\;=\; 0 \;. \tag{4.41b}$$

If we write $\Lambda = \Omega P^{-1} + uP_\zeta P^{-2}$, as in eq.(4.5), then Λ is independent of u, and the transformation $(v,u) \to (r,s)$ of eq.(4.3) transforms the metric to that of eq.(4.4), with the P defined in eq.(4.40) being precisely that in eq.(4.4) and eq.(4.5). The mass function, m, is a constant,

$$m = -2MP^3 = \mu_0 \quad , \tag{4.42}$$

as is the invariant $K_1 = PP_{\zeta\bar{\zeta}} - P_\zeta P_{\bar{\zeta}}$.

The remaining equations to be solved are eqs(F.E.II,D.IV), and the equation for Λ in eq.(4.5d). From eq.(3.31d), and the known results, J can be expressed as

$$J = 2i\delta_\zeta/P \quad , \tag{4.43}$$

and so eq.(D.IV) becomes

$$D(J/M) = 2iPQ_{\zeta\zeta} = 0 \,. \tag{4.44}$$

Since Q is real, it follows that Q is a real bilinear function of ζ and $\bar{\zeta}$. Using the commutator identities, we find that eq.(F.E.II) is just

$$M - \bar{M} = -D\bar{D}\Delta + \bar{D}(\Delta\dot{\Omega}) + D(\Delta\dot{\bar{\Omega}}) + \Delta(\bar{D}\dot{\Omega} - \dot{\Omega}\dot{\bar{\Omega}}) \quad , \tag{4.45}$$

which gives the following relationship between the constants in P and Q, and μ_0,

$$\mathrm{Im}(\mu_0) = QP_{\zeta\bar{\zeta}} - P_\zeta Q_{\bar{\zeta}} - Q_\zeta P_{\bar{\zeta}} + PQ_{\zeta\bar{\zeta}} \,. \tag{4.46}$$

The right hand side is a constant, as may be shown by introducing a homogeneous "spinor" notation. Defining the spinor (ζ^0,ζ^1) through $\zeta = \zeta^1/\zeta^0$, gives

$$P = \mathcal{P}_{A\dot{A}}\zeta^{A}\zeta^{\dot{A}}/|\zeta^{0}|^{2} = \mathcal{P}_{1\dot{1}}\zeta\overline{\zeta} + \mathcal{P}_{0\dot{1}}\overline{\zeta} + \mathcal{P}_{1\dot{0}}\zeta + \mathcal{P}_{0\dot{0}}\,, \tag{4.47a}$$

$$Q = \mathcal{Q}_{A\dot{A}}\zeta^{A}\zeta^{\dot{A}}/|\zeta^{0}|^{2} = \mathcal{Q}_{1\dot{1}}\zeta\overline{\zeta} + \mathcal{Q}_{0\dot{1}}\overline{\zeta} + \mathcal{Q}_{1\dot{0}}\zeta + \mathcal{Q}_{0\dot{0}}\,, \tag{4.47b}$$

so that P and Q correspond to the hermitean spinors $\mathcal{P}_{A\dot{A}}$ and $\mathcal{Q}_{A\dot{A}}$ respectively. The derivatives of P may be written as

$$P_{\zeta} = \mathcal{P}_{1\dot{A}}\zeta^{\dot{A}}/\zeta^{\dot{0}}\,, \quad P_{\zeta\overline{\zeta}} = \mathcal{P}_{1\dot{1}}\,, \tag{4.48}$$

and so eq.(4.46) is

$$\begin{aligned}
\mathrm{Im}(\mu_0) &= (\mathcal{P}_{1\dot{1}}\mathcal{Q}_{A\dot{A}} - \mathcal{P}_{1\dot{A}}\mathcal{Q}_{\dot{1}A} - \mathcal{Q}_{1\dot{A}}\mathcal{P}_{\dot{1}A} + \mathcal{P}_{A\dot{A}}\mathcal{Q}_{1\dot{1}})\zeta^{A}\zeta^{\dot{A}}/|\zeta^{0}|^{2}\,, \\
&= 2(\mathcal{Q}_{\dot{A}[A}\mathcal{P}_{1]\dot{1}} - \mathcal{Q}_{\dot{1}[A}\mathcal{P}_{1]\dot{A}})\,\zeta^{A}\zeta^{\dot{A}}/|\zeta^{0}|^{2}\,, \\
&= 2(\mathcal{Q}_{\dot{0}[0}\mathcal{P}_{1]\dot{1}} - \mathcal{Q}_{\dot{1}[0}\mathcal{P}_{1]\dot{0}})\,, \\
&= \mathcal{P}_{1\dot{1}}\mathcal{Q}^{1\dot{1}} + \mathcal{P}_{1\dot{0}}\mathcal{Q}^{1\dot{0}} + \mathcal{P}_{0\dot{1}}\mathcal{Q}^{0\dot{1}} + \mathcal{P}_{0\dot{0}}\mathcal{Q}^{0\dot{0}}\,, \\
&= \mathcal{P}_{A\dot{A}}\mathcal{Q}^{A\dot{A}}\,,
\end{aligned} \tag{4.49}$$

where spinor indices are raised and lowered with ϵ^{AB}, ϵ_{AB}. This proves our assertion that the right hand side of eq.(4.46) is a constant. Clearly, we also have

$$2K_1 = \mathcal{P}_{A\dot{A}}\mathcal{P}^{A\dot{A}}\,, \qquad 2K_2 = \mathcal{Q}_{A\dot{A}}\mathcal{Q}^{A\dot{A}}\,. \tag{4.50}$$

We shall eventually prove Lemma 1N by using the fact that any Killing vector must be an infinitesimal generator of the coordinate group .

The coordinate transformations which preserve the form of the $\mathcal{Q}$(1N) metric of eq.(4.4) are shown in [K.D.] to be

$$\zeta' = \Phi(\zeta)\,, \quad r' = c_0 r\,, \quad s' = c_0^{-1}(s + A(\zeta,\overline{\zeta}))\,, \tag{4.51}$$

under which

$$P' = |\Phi_\zeta| c_0 P , \quad Q' = |\Phi_\zeta| c_0{}^2 Q , \tag{4.52a}$$

$$\Lambda' = (\Lambda - A_\zeta) / c_0 \Phi_\zeta , \quad m' = c_0{}^3 m , \tag{4.52b}$$

where c_0 is an arbitrary real constant, and Φ is an analytic function of ζ. Since eq.(4.31) must be preserved, it follows from eq.(4.15a) that

$$2(\log \Phi_\zeta)_{\zeta\zeta} = [(\log \Phi_\zeta)_\zeta]^2 . \tag{4.53}$$

The complete solution of this is the bilinear transformation

$$\zeta' = a\zeta + b / (c\zeta + d) , \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 , \tag{4.54}$$

which may be used to simplify the form of P and Q. The transformation properties of $\mathcal{P} = \mathcal{P}_{A\dot{A}} \zeta^A \zeta^{\dot{A}}$ and $\mathcal{Q} = \mathcal{Q}_{A\dot{A}} \zeta^A \zeta^{\dot{A}}$, under eq.(4.54), follow from eq.(4.47) and eq.(4.52). Since $\Phi_\zeta = (ad - bc) / (c\zeta + d)^2 = (\zeta^0 / \zeta^{0'})^2, \mathcal{P}' = \mathcal{P}$ and $\mathcal{Q}' = \mathcal{Q}$.
From eq.(4.54), $\zeta^{A'} = S^{A'}{}_A \zeta^A$, and so $\mathcal{P}_{A'\dot{A}'} = S^A{}_{A'} S^{\dot{A}}{}_{\dot{A}'} \mathcal{P}_{A\dot{A}}$, where

$$S^{A'}{}_A = \begin{pmatrix} d & c \\ b & a \end{pmatrix} , \quad S^A{}_{A'} = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix} .$$

In particular, $\mathcal{P}_{A\dot{A}}$ may be diagonalized by using eq.(4.54).

Any additional Killing vector, K, may be shown from eq.(4.51) to have the form given in eq.(4.9) (†). From eq.(4.42) and eq.(4.11b), K must commute with ∂_s. The Killing equations for K, eq.(4.10) and eq.(4.9), are therefore

$$KP = \mathrm{Re}(\alpha_\zeta) P , \tag{4.55a}$$

(†) see [K.D.].

$$K\delta = 0 \,, \tag{4.55b}$$

$$K = \alpha\partial_\zeta + \bar{\alpha}\partial_{\bar{\zeta}} + T\partial_s \,. \tag{4.55c}$$

Since $(P^2\delta_\zeta)_\zeta = (PQ_\zeta - QP_\zeta)_\zeta = 0$, we have

$$\delta_\zeta = \bar{\chi}(\bar{\zeta})P^{-2} \,, \tag{4.56}$$

where χ is an analytic function of ζ. Eq. (4.55b) may be written as

$$\alpha\delta_\zeta + \bar{\alpha}\delta_{\bar{\zeta}} = 0 \,, \tag{4.57}$$

and so eq. (4.56) and eq. (4.57) assert the existence of a real constant, K_0, with

$$\chi = iK_0\,\alpha(\zeta) \,. \tag{4.58}$$

From eq. (4.56),

$$P^2\delta_{\bar{\zeta}} = iK_0\alpha(\zeta) \,. \tag{4.59}$$

Since any Killing vector is determined up to a multiplicative factor, eq. (4.59) effectively defines the Killing vector in eq. (4.55c), provided that $K_0 \neq 0$. For this case, $\alpha = -iP^2\delta_{\bar{\zeta}}/K_0$, eq. (4.55a) and eq. (4.55b) are satisfied, and so there are precisely two commuting Killing vectors.

The equation to be solved for Λ, eq. (4.5d), may be written as $\bar{\Lambda}_\zeta = \hat{f}_{\zeta\bar{\zeta}} + i\delta P^{-2}$, where $\hat{f}_{\zeta\bar{\zeta}} = \mathrm{Re}(\Lambda_\zeta)$, and $\hat{f}$ is a real function of ζ and $\bar{\zeta}$. The general solution for $\bar{\Lambda}$ is then $\bar{\Lambda} = f_{\bar{\zeta}} + i\int[\,(\delta P^{-2})|_{\bar{\zeta} = \text{constant}}]\,d\zeta$, where f equals $\hat{f}$, modulo an arbitrary function of $\bar{\zeta}$. Under the coordinate transformation $u \to u + f$, $\bar{\Lambda} \to \bar{\Lambda} - f_{\bar{\zeta}}$ and so the $f_{\bar{\zeta}}$ term in $\bar{\Lambda}$ may be eliminated. The general solution for Λ may be taken as any particular solution, and, since $(\delta^2/\alpha)_{\bar{\zeta}} = 2iK_0\delta P^{-2}$, we may therefore write

$$\Lambda = \delta^2/2K_0\alpha = i\delta^2/2P^2\delta_\zeta \ , \tag{4.60}$$

provided $\delta_{\bar{\zeta}} = 0$. From eq.(4.11) and eq.(4.60),

$$- T_\zeta = (\alpha\Lambda)_\zeta + \bar{\alpha}\Lambda_{\bar{\zeta}} = (\alpha\Lambda - \overline{\alpha\Lambda})_\zeta = 0 \ , \tag{4.61}$$

so that T is an arbitrary constant. By adding a multiple of the other Killing vector onto K, this constant can be removed, so that T may be assumed to be zero. Finally, we may use eq.(4.54) to set K_0 in eq.(4.60) equal to unity.

We shall now consider the $\delta_\zeta = 0$ case, and prove Lemma 1N. When δ is a constant, the two real bilinear forms P and Q are proportional. Equation (4.54) may be used to set

$$P = \zeta\bar{\zeta} + K_1 \ . \tag{4.62}$$

Any <u>finite</u> symmetry must be a bilinear transformation in ζ, and if $t \to \psi(\zeta,t)$ is the one parameter subgroup corresponding to α, then

$$\alpha(\zeta) = \partial_t\psi|_{t=0} \quad , \tag{4.63a}$$

where

$$\psi(\zeta,t) = a\zeta + b\,/\,(c\zeta + d) \ , \tag{4.63b}$$

and a, b, c, d, are all functions of t. Since $\psi(\zeta,0) = \zeta$, we have $a(0) = 1$, $b(0) = 0$, $c(0) = 0$, $d(0) = 1$, and so

$$\alpha(\zeta) = \ - c_t\zeta^2 + (a_t + d_t)\zeta + b_t \ , \tag{4.64}$$

where $q_t = q(t)_{,t}|_{t=0}$, for q = a, b, c, d. If we use the notation of eq.(4.47), α corresponds to a symmetric spinor, $\mathcal{A}_{CD}$,

$$\alpha = (\mathscr{A}_{CD} / |\zeta^0|^2) \; \zeta^C \zeta^D . \tag{4.65}$$

Substituting eq.(4.64) into eq.(4.10) gives as the general solution for α,

$$\alpha = \alpha_0 \zeta^2 + ie_0 \zeta + K_1 \bar{\alpha}_0 , \tag{4.66}$$

where α_0 and e_0 are arbitrary constants, e_0 being real. The existence of this four parameter group of symmetries finally proves Lemma 1N. In particular, if $\delta_\zeta = 0$, $\alpha = i\zeta$ can always be chosen as a preferred solution to the Killing equation, and then

$$\delta = \text{constant} \Rightarrow P = \zeta\bar{\zeta} + K_1 , \; \alpha = i\zeta , \; \Lambda = i\delta/\zeta(\zeta\bar{\zeta} + K_1) . \tag{4.67}$$

To end this chapter, we shall briefly explain the use of the terms radiating and non-radiating. In our coordinate system, Trim and Wainwright[14] define a metric to be radiating if the scalars ψ_2, ψ_3, ψ_4 in eq.(3.33) contain any terms involving ρ or ρ^2, and non-radiating otherwise. These definitions proved cumbersome in our discussions, so they were replaced by the non-equivalent ones of this chapter.

In the Type D spaces, however, these definitions are equivalent, and the $\bar{D}\partial_u D\Omega = 0$ case gives, from eq.(3.33) and the results of this chapter,

$$\psi_2 = -2M\rho^3 ,$$

$$\psi_3 = 3MJ^2\rho^4 ,$$

$$\psi_4 = -2MJ^2\rho^5 ,$$

$$J = 2i\delta_\zeta / P ,$$

so that $\psi_3 = \psi_4 = 0$ when δ is a constant. Since the only empty, diverging, algebraically special spaces with four Killing vectors are Type D,(†), the $\omega^{0\dot{0}}$ in eq.(3.27a) is another double Debever vector iff the space has four Killing vectors.

(†) See [K.D.].

CHAPTER V

METRICS OF TYPE $\mathcal{A}(2,R)$ AND $\mathcal{D}(2,R)$

The metrics of Type $\mathcal{D}(2,R)$ are all independent of ζ and $\bar{\zeta}$, and so we shall have a close look at the larger class, $\mathcal{A}(2,R)$, of algebraically special spaces with the same symmetries. Our immediate objective is to find when an $\mathcal{A}(2,R)$ becomes a $\mathcal{D}(2,R)$, and then to solve for all $\mathcal{D}(2,R)$ metrics. For $\mathcal{A}(2,R)$ spaces, the field equations, eq.(3.34), reduce to ordinary differential equations, (since $D = -\Omega\partial_u$), and these can be partially integrated. From eq.(3.34a),

$$M = -\tfrac{1}{2}\mu_0\bar{\Omega}^{-3} \quad , \tag{5.1}$$

where μ_0 is a complex constant. Eq.(3.34c) can now be written as

$$\frac{d}{du}\left(\bar{\Omega}^2\, H\right) = -3\mu_0\bar{\Omega}^{-3}\bar{\Omega}_{,u} \;, \qquad \text{and so}$$

$$2H = 2\bar{D}\partial_u D\Omega = 3\mu_0\bar{\Omega}^{-4} - \nu_0\bar{\Omega}^{-2} \quad , \tag{5.2}$$

where ν_0 is another complex constant. Since

$$\bar{D}\bar{D}D\Omega = -\bar{D}(\bar{\Omega}\partial_u D\Omega) = -\bar{\Omega}H - (\bar{D}\bar{\Omega})\partial_u D\Omega \;,$$

the last field equation, eq.(3.34b), can be written as

$$\mathrm{Im}[\,2(\bar{D}\bar{\Omega})\partial_u\bar{D}\bar{\Omega} - \mu_0\bar{\Omega}^{-3} + \nu_0\bar{\Omega}^{-1}] = 0 \;. \tag{5.3}$$

Eq.(5.2) and eq.(5.3) can be simplified by the substitution

$$\Omega = g^{-\frac{1}{2}}\, e^{i\theta} \quad , \tag{5.4}$$

together with a change of variable from u to a variable x defined by

$$dx = g^{3/2}\, du\ , \qquad \partial_u = g^{-3/2}\, \partial_x\ . \tag{5.5}$$

The field equations, eqs(3.34), then become

$$\dddot{\theta} g - \ddot{\theta}\dot{g} + (2\dot{\theta}^3 - \dddot{\theta})g = -\,\mathrm{Im}\,(\mu_0 e^{3i\theta})\ , \tag{5.6}$$

$$\dddot{g} + 6\dot{\theta}^2\,\dot{g} + 12\dot{\theta}\ddot{\theta}g = \mathrm{Re}\,(-\,3\mu_0 e^{3i\theta} + \nu_0 g^{-1}\, e^{i\theta})\ , \tag{5.6a}$$

$$(2\dot{\theta}^3 + \dddot{\theta})g = \mathrm{Im}(\nu_0 g^{-1}\, e^{i\theta})\ , \tag{5.7}$$

where a dot denotes differentiation with respect to x. These equations are not independent, since

$$\frac{d}{dx}\,(5.6) + g^{-1}\,\frac{d}{dx}\,(5.7) = \dot{\theta}\,\ (5.6a)\ ,$$

and so we shall ignore eq.(5.6a) when $\dot{\theta}$ is non-zero.

When the metric is Type D, eq.(5.2) can be simplified, with the aid of eq.(4.13) and eq.(5.1), to $\nu_0 = 0$. Conversely, if ν_0 is zero, then from eq.(4.13), $(HM^{-4/3})$ is a constant, and so eq.(3.37a) and eq.(3.37b) are satisfied. The last two Type D equations, eq.(3.37c) and eq.(3.37d) follow from eq.(5.6) and eq.(5.7), proving

<u>Theorem IIR</u>: A space of Type $\mathfrak{A}(2,R)$ is Type D iff ν_0 is zero; i.e. iff $H\bar{\Omega}^4$ is constant.

For Type $\mathcal{D}(2,R)$ spaces, eq.(5.7) can be integrated, giving

$$e^{-2i\theta}\ (\dot{\theta}^2 - i\ddot{\theta}) = \tfrac{1}{2}\lambda_0\ ,$$

where λ_0 is a constant, and so

$$\dot{\theta}^2 = \tfrac{1}{2}\,\mathrm{Re}\,(\lambda_0\, e^{2i\theta})\ ,\quad \ddot{\theta} = -\,\tfrac{1}{2}\,\mathrm{Im}(\lambda_0 e^{2i\theta})\ . \tag{5.8}$$

The remaining coordinate freedom is

$$\zeta' = q_0\zeta + q_1 \,, \tag{5.9a}$$

$$u' = |q_0|(u + h_0) \,, \tag{5.9b}$$

$$\Omega' = \Omega|q_0|/q_0 \,, \tag{5.9c}$$

where q_0, q_1 and h_0 are all constants. From eq.(5.4) and eq.(5.5),

$$g' = g \,, \quad \theta' = \theta - \arg(q_0) \,, \quad x' = |q_0|(x - x_0) \,, \tag{5.9d}$$

x_0 being another constant, and so $\lambda' = \lambda_0 \bar{q}_0^{-2}$. Providing λ_0 is non-zero, it can be transformed to any required value. In hindsight(†), the best choice is possibly $\lambda_0 = -i$, but we shall follow Kinnersley and choose $\lambda_0 = 1$. Eq.(5.8) may then be written as

$$\dot{\theta}^2 = \tfrac{1}{2}\cos 2\theta \,, \quad \ddot{\theta} = -\tfrac{1}{2}\sin 2\theta \,, \tag{5.10}$$

which have as solution

$$e^{i\theta} = \mathrm{dn}(x) + \frac{i}{\sqrt{2}}\,\mathrm{sn}(x) \,, \quad \dot{\theta} = \frac{1}{\sqrt{2}}\,\mathrm{cn}(x) \,, \tag{5.11}$$

where the constant of integration has been eliminated by the coordinate transformation $x \to x - x_0$. The functions in eq.(5.11) are the Jacobian elliptic functions of modulus $(1/\sqrt{2})$. The only remaining coordinate freedom is $\zeta' = \zeta + q_1$, which corresponds to the complex Killing vector, ∂_ζ, being trivially reparametrized, and this can lead to no more simplications in the metric as we always have this freedom.

When $\dot{\theta} \neq 0$, eq.(5.6a) can be ignored and eq.(5.6) solved for g. The complete solution is

(†) See Appendix A.

$$g(x) = 2\,\mathrm{Re}\,(a_0 e^{2i\theta} + b_0 \dot{\theta}\, e^{i\theta}) \,, \tag{5.12}$$

where a_0 and b_0 are arbitrary complex constants, and the mass parameter, μ_0, is then given by

$$\mu_0 = \tfrac{1}{2} i\, \bar{b}_0 \,. \tag{5.13}$$

When $\dot{\theta} = 0$, or equivalently $\lambda_0 = 0$, we have $\mathrm{Im}(\bar{D}\Omega) = 0$ and so $\Delta = 0$, and the complex-divergence ρ is real. This is the necessary and sufficient condition for $\omega^{1\dot{1}}$ to be hyper-surface orthogonal. The field equations, eq.(5.6), become

$$\dddot{g} + \mathrm{Re}\,(3\mu_0 e^{3i\theta_0}) = 0 \,, \quad \mathrm{Im}(\mu_0 e^{3i\theta_0}) = 0 \,. \tag{5.14}$$

The transformation $\theta' = \theta - \arg(q_0)$ can be used to make $\theta_0 = 0$, so that $\Omega = g^{-\frac{1}{2}}$ and μ_0 is real. From eq.(5.6b),

$$g = -\tfrac{1}{2}\mu_0 x^3 + g_2 x^2 + g_1 x + g_0 \,,$$

where $\mu_0 \neq 0$, g_0, g_1, and g_2 are real constants. The transformation $x' = |q_0|(x - x_0)$ can be used to set $\mu_0 = -2$, and to eliminate the quadratic term. Finally, the canonical form for g is

$$g = x^3 + a_0 x + b_0 \,, \tag{5.15}$$

and we will see later that this gives the C- metric of Ehlers and Kundt.

The metrics resulting from eq.(5.11), eq.(5.12) and eq.(5.15) contain all $\mathscr{D}(2,R)$ spaces. The main aim for the remainder of this chapter is to find canonical forms for the corresponding metrics.

The $\mathfrak{A}(2,R)$ metrics can be transformed to

$$\begin{aligned} ds^2 = & -2g\,\Sigma\,[\,d\zeta^* d\bar{\zeta}^* + dx^2/4g^2] \\ & + 2[\,e^{i\theta}\, d\zeta^* + e^{-i\theta}\, d\bar{\zeta}^*]\;[dR + p\,dx + \sigma d\zeta^* + \bar{\sigma} d\bar{\zeta}^*] \end{aligned} \;, \tag{5.16}$$

where

$$\Sigma = R^2 + \dot{\theta}^2 , \quad p = \tfrac{1}{2}(R^2 + 3\dot{\theta}^2) \; , \tag{5.17a}$$

$$\sigma e^{-i\theta} = \mathrm{Re}[-\mu_0 e^{3i\theta}/(R+i\theta)] + \tfrac{1}{2}(R\dot{g} - \ddot{g})$$

$$- g\dot{\theta}^2 - iRg\dot{\theta} \, , \tag{5.17b}$$

and the coordinates are defined by

$$d\zeta^* = d\zeta + dx/2ge^{i\theta} , \quad R = vg^{-\frac{1}{2}} . \tag{5.18}$$

We shall now show that this metric can be simplified, if $\nu_0 = 0$, by using some results due to Papapetrou[7]. We shall follow his notation and write the metric as

$$ds^2 = g_{\mathfrak{a}\mathfrak{b}} dx^{\mathfrak{a}} dx^{\mathfrak{b}} + 2g_{\mathfrak{a}a} dx^{\mathfrak{a}} dx^a + g_{ab} dx^a dx^b , \tag{5.19}$$

where $\mathfrak{a}, \mathfrak{b} \in \{1,2\}$; $a,b \in \{3,4\}$; and $g_{\alpha\beta}$ is a function of the $\{x^{\mathfrak{a}}\}$; i.e. the $\{\partial_a\}$ are Killing vectors. In eq.(5.16), $x^{\alpha} = (R,x,\zeta,\bar{\zeta})$. We will say that the metric in eq.(5.19) is quasi-diagonalizable if the cross-terms, $g_{\mathfrak{a}a}$, can be eliminated. This can only be done by a transformation of the type

$$x^a = x^{a'} + f^a(x^{\mathfrak{a}}) , \quad x^{\mathfrak{a}'} = x^{\mathfrak{a}} , \tag{5.20}$$

for which

$$g_{\mathfrak{a}'a'} = g_{\mathfrak{a}a} + g_{ab} f^b{}_{,\mathfrak{a}} , \tag{5.21}$$

and so the equation $g_{\mathfrak{a}'a'} = 0$ can be solved for the df^a,

$$df^a = -h^{ab} g_{b\mathfrak{a}} dx^{\mathfrak{a}} , \tag{5.22a}$$

$$g_{ac} h^{cb} = \delta^b_a . \tag{5.22b}$$

The metric is therefore q.d. iff the right hand side of eq.(5.22a) is a perfect differential.

For the $\mathcal{O}\mathcal{L}(2,R)$ metrics, eq.(5.22) gives

$$d\eta - d\zeta^* = -e^{-i\theta}[g\Sigma - \sigma e^{-i\sigma} + \bar{\sigma}e^{i\theta}].[dR + pdx]/G, \qquad (5.23)$$

where $\eta = \zeta^{*\prime}$, and G is the determinant of the two metric, g_{ab},

$$G = -g^2\Sigma^2 + 4g\Sigma\,\mathrm{Re}[\sigma e^{-i\theta}] + 4[\mathrm{Im}(\sigma e^{-i\theta})]^2. \qquad (5.24)$$

Such a metric is q.d. iff $d(\eta - \zeta^*)$ is a perfect differential. It is shown in Appendix (B) that this is so iff ν_0 is zero, which proves

<u>Theorem IIIR</u>: A metric of Type $\mathcal{O}\mathcal{L}(2,R)$ is q.d. iff it is of Type D.

When η exists, the metric can be written as

$$(ds)^2 = -2g\Sigma[d\eta d\bar{\eta} + (dx/2g)^2 + (dR+pdx)^2/G]$$
$$+ 2(e^{i\theta}d\eta + e^{-i\theta}d\bar{\eta}) \quad (\sigma d\zeta + \bar{\sigma}d\bar{\eta}). \qquad (5.25)$$

It is instructive to consider first the $\dot{\theta} = 0$ case, which leads to the C metric.

When $\dot{\omega}^{1i}$ is hyper-surface orthogonal, $\dot{\theta} = 0$, and we can choose $e^{i\theta} = 1$, so that Ω is real. Eq.(5.23) becomes

$$d(\eta - \zeta^*) = -(gR^4/2G)\,d(x - 2/R), \qquad (5.26)$$

and so we shall replace the coordinate R by $y = x - 2/R$. Now $\Sigma = R^2$, and from eq.(5.24);

$$G/R^4g = -g(x) + 4R^{-2}\,\mathrm{Re}(\sigma e^{-i\theta}) \qquad (5.27a)$$
$$= -g(x) + 0(R^{-1}). \qquad (5.27b)$$

Since the R.H.S. of this expression must be a function of y, and as g is a polynomial, it follows that

$$G = - R^4 g(x)\ g(y)\ . \tag{5.28}$$

Of course, this can also be proved by direct calculation. It is now fairly simple to show that

$$d\eta = d\zeta + dx/2g(x) + dy/2g(y)\ . \tag{5.29}$$

If we write $2\eta = \eta_1 + i\eta_2$, where the η_1, η_2 are real, then the metric may be written as

$$ds^2 = \frac{-2}{(x-y)^2}\,[\,g(x)d\eta_1^2 + dx^2/g(x) + g(y)d\eta_2^2 - dy^2/g(y)]\ . \tag{5.30}$$

The first p.n.v. is given by $g^{\frac{1}{2}}(du + \Omega d\zeta + \bar{\Omega} d\bar{\zeta}) = d\eta_1 - dy/g(y)$. The metric is invariant under $d\eta \rightarrow -d\eta$, and so the second p.n.v. is got from the first by changing the sign of $d\eta$. We shall denote the two p.n.v.'s by $K\pm$,

$$K\pm = d\eta_1 \pm dy/g(y)\ . \tag{5.31}$$

The curvature invariant, ψ_2, is given by

$$\psi_2 = \tfrac{1}{8}\mu_0\,(x-y)^3\ , \tag{5.32}$$

and so the curvature tensor is regular for all finite values of x and y. This is the C metric of Ehlers and Kundt[22].

The canonical form, eq.(5.30), for the C metric was found by replacing the coordinate R by a new variable, y, whose differential was proportional to $(dR + pdx)$. For the general case of eq.(5.23), $\dot{\theta} \neq 0$, we seek a variable y having the same property. From eq.(5.23),

$$d(\zeta^* - \eta) = (ge^{-i\theta}/G)\ (\Sigma + iA)\ dy/y_R\ , \tag{5.33}$$

where

$$A = 2\mathrm{Im}(\bar{\sigma}e^{i\theta}/g) = 2R\dot{\theta} + \sin 2\theta\,. \tag{5.34}$$

Eq.(5.33) may be written as

$$d(\zeta^* - \eta) = (C_1+iC_2)\,dy\,, \tag{5.35}$$

where C_1 and C_2 are real functions of y, and so the ratio C_2/C_1, given by

$$C_2/C_1 = -\tan[\theta - \tan^{-1}(A/\Sigma)]\,, \tag{5.36a}$$

must itself be a function of y. Finally, y can be defined by setting

$$C_2/C_1 = -\tan\phi(y)\,, \tag{5.36b}$$

and requiring that

$$\left(\frac{d\phi}{dy}\right)^2 = \tfrac{1}{2}\cos 2\phi\,. \tag{5.36c}$$

This will always work for $\dot{\theta} \neq 0$, since C_2/C_1 is only constant when $2\theta = 0.\mathrm{mod}(\pi)$, and C_1 is non-zero.

We shall now express the metric of eq.(5.25) in terms of coordinates x and y. From eq.(5.36),

$$\Sigma + iA = (\Sigma^2+A^2)^{\frac{1}{2}}\, e^{i(\theta-\phi)}\,. \tag{5.37}$$

This immediately implies that

$$e^{2i(\theta-\phi)} = (\Sigma+iA)/(\Sigma-iA)\,, \tag{5.38}$$

from which we calculate

$$(\Sigma^2+A^2)\,d\phi = (\Sigma^2+A^2)\,d\theta + (A\,d\Sigma - \Sigma\,dA)\,. \tag{5.39}$$

Substituting in the known values for A and Σ gives

$$(\Sigma^2 + A^2)\,d\phi = 2(R^2\dot{\theta} + R\sin 2\theta - \dot{\theta}^3)\ (dR + pdx)\ . \tag{5.40}$$

The following set of equations may be found by defining $\dot{\phi} = (d\phi/dy)$, and using eq.(5.36), eq.(5.17a) and eq.(5.10),

$$\begin{aligned} 2\ (\Sigma^2 + A^2)\dot{\phi}^2 &= (\Sigma^2 + A^2)\cos 2\phi \quad , \\ &= (\Sigma^2 + A^2)\ (1 - \tan^2\phi)\ /\ (1 + \tan^2 2\phi)\ , \\ &= (\Sigma^2 - A^2)\cos 2\theta + 2A\Sigma\sin 2\theta \quad , \\ &= 2[R^2\dot{\theta} + R\sin 2\theta - \dot{\theta}^3]^2 \quad , \end{aligned}$$

from which we conclude that

$$\sqrt{(\Sigma^2 + A^2)}\,\dot{\phi} = (R^2\dot{\theta} + R\sin 2\theta - \dot{\theta}^3)\ . \tag{5.41}$$

By comparing this with eq.(5.40) we see that

$$(\Sigma^2 + A^2)\ \dot{\phi}\ dy = 2\ (R^2\dot{\theta} + R\sin\theta - \dot{\theta}^3)\ (dR + pdx)\ , \tag{5.42a}$$

$$= 2\sqrt{(\Sigma^2 + A^2)}\,\dot{\phi}\ (dR + pdx)\ , \tag{5.42b}$$

and so

$$dR + pdx = \tfrac{1}{2}\sqrt{(\Sigma^2 + A^2)}\,dy\ . \tag{5.43}$$

Substituting eq.(5.43), eq.(5.34) and eq.(5.36) into eq.(5.23) then gives

$$d(\zeta^* - \eta) = [G/g(\Sigma^2 + A^2)]^{-1}\quad \tfrac{1}{2}e^{-i\phi}\,dy\ . \tag{5.44}$$

The expression in square brackets, which is linear in the complex integration constants, a_0 and b_0, must be a function of y. From the corresponding result for the C metric it was conjectured to be proportional to g(y), so that

$$G(x,R) / g(x) = -g(y)\,(\Sigma^2 + A^2)\,. \tag{5.45}$$

The easiest way to prove this is to substitute for g(y), using eq.(5.12) and eq.(5.36),

$$(\Sigma^2 + A^2)\,g(y) = 2(\Sigma^2 + A^2)\,\mathrm{Re}[\,a_0 e^{2i\phi} + b_0 e^{i\phi}\dot{\phi}\,]\,, \tag{5.46a}$$

$$= 2\mathrm{Re}\,[\,a_0 e^{2i\theta}(\Sigma - iA)^2$$

$$+ b_0 e^{i\theta}(\Sigma - iA)(R^2\dot{\theta} + R\sin 2\theta - \dot{\theta}^3)]\,. \tag{5.46b}$$

Eq.(5.45) can then be verified by comparing it with eq.(5.46b).

The function σ still has to be expressed as a function of x and y. From eq.(5.24), eq.(5.34) and eq.(5.45),

$$4\,\Sigma\,\mathrm{Re}\,(\sigma e^{-i\theta}) = g(x)\;(\Sigma^2 - A^2) - g(y)\;(\Sigma^2 + A^2)\,. \tag{5.47}$$

Using eq.(5.47), eq.(5.34) and eq.(5.36) we find that

$$\sigma = e^{i\theta}\,[\,\mathrm{Re}(\sigma e^{-i\theta}) + i\,\mathrm{Im}(\sigma e^{-i\theta})]\,, \tag{5.48a}$$

$$= e^{i\theta}\,[\,g(x)\;(\Sigma - iA)^2 - g(y)\;(\Sigma^2 + A^2)]\,/\,4\Sigma\,, \tag{5.48b}$$

$$= \Sigma\, e^{-i\theta}\sec^2(\theta - \phi)\;[\,g(x)e^{2i\phi} - g(y)e^{2i\theta}]\,/\,4\,. \tag{5.48c}$$

The equation for η can be written as

$$d\eta = d\zeta + dx/2e^{i\theta}g(x) + dy\,/\,2e^{i\phi}\,g(y)\,, \tag{5.49}$$

and the metric, eq.(5.25), then takes the simple form

$$ds^2 = \tfrac{1}{2}\Sigma[\,\sec^2(\theta - \phi)\;g(x)\,(e^{i\phi}d\eta - e^{-i\phi}d\bar{\eta})^2 - dx^2\,/\,g(x)$$

$$- \sec^2(\theta - \phi)\;g(y)\;(e^{i\theta}d\eta + e^{-i\theta}d\bar{\eta})^2 + dy^2/g(y)]\,, \tag{5.50}$$

$$\Sigma = \tfrac{1}{2}\operatorname{cosec}^2(\theta-\phi)\,[\cos 2\theta + \cos 2\phi + 2\cos(\theta-\phi)\sqrt{\cos 2\theta \cos 2\phi}] \,. \tag{5.51}$$

The last relationship follows from eq. (5.36), which may be written as

$$R = (\sqrt{\cos 2\theta} + \sqrt{\cos 2\phi}) \;/\; \sqrt{2}\sin(\theta-\phi) \,. \tag{5.52}$$

The first p.n.v. is given by

$$2\,\mathrm{Re}[e^{i\theta} d\zeta^*] = 2\,\mathrm{Re}\,(e^{i\theta} d\eta) + \cos(\theta-\phi)\,dy/g(y) \,, \tag{5.53}$$

and so the two p.n.v.'s, $K\pm$, are

$$K\pm = \cos(\theta-\phi)\,dy/g(y) \pm 2\,\mathrm{Re}\,(e^{i\theta} d\eta) \,. \tag{5.54}$$

The similarities between eq. (5.30) and eq. (5.50) are obvious. The function, g, appears to be more complicated in Kinnersley's metric, eq. (5.50), since it involves elliptic functions, but it can be reduced to polynomial form by defining (see Appendix C)

$$X^2 = \tan(\theta + \tfrac{\pi}{4}),\quad Y^2 = \tan(\phi + \tfrac{\pi}{4}),\quad \eta = \tfrac{1}{2}(\eta_2 + i\eta_1)\,e^{i\pi/4} \,. \tag{5.55}$$

The metric then becomes

$$ds^2 = -2(X-Y)^2\Big\{[\tilde{g}(X)(d\eta_1 + Y^2 d\eta_2)^2 + \tilde{g}(Y)(d\eta_2 - X^2 d\eta_1)^2] \;/(1 + X^2Y^2)$$

$$+ [dX^2/\tilde{g}(X) - dY^2/\tilde{g}(Y)]\;(1 + X^2Y^2)\Big\} \,, \tag{5.56}$$

where $\tilde{g}$ is a quartic polynomial,

$$\tilde{g}(X) = c_0(1 - X^4) + c_1X + c_2X^2 + c_3X^3 \,, \tag{5.57}$$

$$4a_0 = c_1 + 2ic_0 \,,\quad 2b_0 = c_1(1+i) + c_3(1-i) \,, \tag{5.58}$$

$$K\pm = d\eta_2 - X^2\, d\eta_1 \pm (1 + X^2 Y^2)\, dY / \tilde{g}(Y) \; . \tag{5.59}$$

To see that eq.(5.30) is the limit of eq.(5.56), it is necessary to let $(X, Y, \eta_1, \eta_2) \to (qX, qY, \eta_1/q, \eta_2/q)$ in eq.(5.56), so that

$$\tilde{g}(qX) = d_0(1 - q^4 X^4) + d_1 X + d_2 X^2 + d_3 X^3 \; , \tag{5.60}$$

where the $\{d_i\}$ are arbitrary. If we set $q = 0$ in the resultant metric, we get eq.(5.30).

Another simple limit follows from eq.(5.56) by defining $Y = Z^{-1}$, $\tilde{h}(Z) = -Z^4 \tilde{g}(Z^{-1})$, or

$$\tilde{h}(Z) = c_0(1 - Z^4) - c_3 Z - c_2 Z^2 - c_1 Z^3 \; . \tag{5.61}$$

Eq.(5.56) becomes

$$ds^2 = 2(1 - XZ)^{-2} \, [\Sigma\, dX^2 / \tilde{g}(X) + \Sigma\, dZ^2 / \tilde{h}(Z)$$

$$+ \tilde{g}(X)(d\eta_2 + Z^2 d\eta_1)^2 / \Sigma - \tilde{h}(Z)(d\eta_2 - X^2 d\eta_1)^2 / \Sigma] \; , \tag{5.62}$$

$$\Sigma = X^2 + Z^2 \; . \tag{5.63}$$

The Kerr-NUT metric of the next section is obtained by making the substitution

$$(X, Z, \eta_1, \eta_2, c_0, c_1, c_2, c_3) \to (qX, qZ, \eta_1/q, \eta_2/q^3 \; ,$$

$$q^4 c_0, q^3 c_1, q^2 c_2, q^3 c_3) \; , \tag{5.64}$$

and then putting $q = 0$.

The curvature invariant, ψ_2, of eq.(5.62) is

$$\psi_2 = (c_1 - ic_3)[(X - Z) / (1 - iXZ)]^3 / 2\sqrt{2} \; , \tag{5.65}$$

and so the curvature tensor is again regular for all finite values of X and Z.

CHAPTER VI

METRICS OF TYPE $\mathcal{O}\!\!\mathcal{L}(1,N)$ AND $\mathcal{D}(2,N)$

In this chapter we extend the results of Chapter V to the non-radiating spaces, eventually showing the members of $\mathcal{D}(2,N)$ to be quasi-diagonalizable. First, we give the following condition for an $\mathcal{O}\!\!\mathcal{L}(1,N)$ to be a $\mathcal{D}(2,N)$:

<u>Lemma 2N</u>: The non-radiating metric of eq.(4.4) is Type D iff K_1 is a constant, $(P^2\,\delta_\zeta)_\zeta = 0$, and m is a non-zero constant whose imaginary part is defined in eq.(4.8b).

<u>Proof</u>: From the results of Chapter IV, if $\not{g} \in \mathcal{D}(2,N)$, then K_1 and m are constants, and $(P^2\,\delta_\zeta)_\zeta = 0$. We shall now prove the converse. From eq.(4.2), if $P\partial_u$ is a real Killing vector, then

$$\dot{M} = 0 = \dot{\Omega} + P_\zeta / P . \tag{6.1}$$

For a general ζ coordinate we shall therefore have $\Omega = P\Lambda(\zeta,\bar{\zeta}) - u\, P_\zeta/P$, so that $\partial_u D\Omega = -P_{\zeta\zeta}/P$. Substituting this into $\partial_u\partial_u D\Omega$ and $\bar{D}\partial_u D\Omega$ gives

$$\partial_u\partial_u D\Omega = 0\ , \quad H = -K_{1,\zeta} / P^2 . \tag{6.2}$$

Eq.(F.E.I) yields

$$M = m(\zeta)\ P^{-3} . \tag{6.3}$$

Consequently, eq.(D.I) and eq.(D.II) are satisfied if K_1 is a constant. From the definition of J in eq.(3.31d), $JM^{-1} = 2iP^2\,\delta_\zeta/m(\zeta)$, and so eq.(D.IV) also holds if m is a constant and

$(P^2 \delta_\zeta)_\zeta = 0$. A straightforward calculation, using $H = 0$ and $\partial_u D\Omega = -P_{\zeta\zeta}/P$, shows that eq.(D.III) is also satisfied, which proves Lemma 2N.

Before discussing the quasi-diagonalization of the $\mathcal{D}(2,N)$ spaces we shall consider the non-radiating spaces with two commuting Killing vectors; i.e. $\mathcal{O}\mathcal{L}(2,N)$. These spaces have another Killing vector, besides ∂_s, which is defined in eq.(4.9), together with the additional condition that $a_0 = 0$. As we are not considering the Type D subclass we do not have the restriction of eq.(4.54), and the full coordinate group in eq.(4.51) may be used. Under this,

$$\alpha' = \Phi_\zeta \alpha \,, \tag{6.4}$$

where α is defined in eq.(4.9), and so we shall choose Φ such that $2\alpha = 1$, and write

$$2d\zeta = d\phi + id\theta \,, \quad \partial_\zeta = \partial_\phi - i\partial\theta \,, \tag{6.5}$$

where ϕ and θ are both real. We shall now show that the additional coordinate freedom in eq.(4.51) allows us to write our second Killing vector, K, as

$$K = \partial_\phi \,; \tag{6.6}$$

or equivalently, that the T in eq.(4.9a) may be chosen to be a real constant.

First, we note that any particular solution of eq.(4.5d) may be taken as the general solution, since the coordinate transformation, $s' = s + A(\zeta,\bar{\zeta})$, in eq.(4.51) may be used to remove the complementary functions of eq.(4.5d) in exactly the same way as was shown to be possible in Chapter IV. Since the metric is independent of s, the

Killing equations in eq.(4.10) reduce to

$$\partial_\phi P = \partial_\phi Q = \partial_\phi m = 0 , \tag{6.7}$$

and so P, Q and m are functions of θ. The particular solution of eq.(4.5d), $\mathrm{Im}(\bar{\Lambda}_\zeta) = QP^{-3}$, may therefore be chosen with Λ a real function of θ. Then, from eq.(4.11), T is a real constant, which w.l.o.g. may be chosen to be zero.

With this choice of coordinates, the metric for the $\mathcal{OL}(2,N)$ spaces is

$$ds^2 = -(\Sigma/2P^2)[(d\theta)^2 + \{d\phi + (2\delta\dot{P}^2/\Sigma)(ds + \Lambda d\phi)\}^2]$$

$$+ [2dr + (W/\Sigma)(ds + \Lambda d\phi)][ds + \Lambda d\phi] , \tag{6.8}$$

where a dot denotes differentiation with respect to θ, and the metric is independent of ϕ and s. The field equations for this metric follow from eq.(6.5) and eq.(4.8),

$$\text{F.E.I} \;:\; m_\theta = 0 , \tag{6.9a}$$

$$\text{F.E.II} \;:\; \mathrm{Im}(m) = PQ_{\theta\theta} - 2P_\theta Q_\theta + P_{\theta\theta}Q , \tag{6.9b}$$

$$\text{F.E.III} \;:\; K_{1,\theta\theta} = 0 , \tag{6.9c}$$

and eq.(4.5d) may be written as

$$\Lambda_\theta = -QP^{-3} . \tag{6.10}$$

Eq.(6.9c) may be integrated once,

$$K_{1,\theta} = n_1 , \tag{6.11}$$

and a first integral of eq.(6.10) is then

$$P^{-2}(P^2\dot{\delta})_\theta - 2\Lambda n_1 = n_2\,, \tag{6.12}$$

where n_1 and n_2 are real constants. Eq. (6.12) is verified by noting that

$$\begin{aligned} \dot{n}_2 &= -2P^{-2}(\dot{P})^2\dot{\delta} + 2P^{-1}\ddot{P}\,\dot{\delta} + 2P^{-1}\dot{P}\,\ddot{\delta} \\ &\quad + 2\delta n_1 P^{-2} + \dddot{\delta}\,, \\ &= 2P^{-2}\dot{\delta}[P\ddot{P} - (\dot{P})^2 - K_1] = 0\,, \end{aligned} \tag{6.13}$$

where we have used eq. (6.12), eq. (6.10), eq. (6.11), ∂_θ[eq. (6.9b)] and eq. (4.6).

From eq. (6.9a), m is a constant, and clearly n_1 and n_2 are zero iff K_1 is a constant and $(P^2\dot{\delta})_\theta$ is zero. This statement, together with Lemma 2N, proves

Lemma 3N: An $\mathfrak{A}(2,N)$ metric is a $\mathcal{D}(2,N)$ metric iff the constants n_1 and n_2 in eq. (6.11) and eq. (6.12) respectively are zero.

We shall now prove

Lemma 4N: If the metric of eq. (4.4) is Type D, then K_2 is a constant.

Proof: Either Q is zero, in which case K_2 is zero, or Q is non-zero, and the identity

$$((P^2\delta_\zeta)_\zeta / PQ)_{\bar{\zeta}} = Q^{-2}K_{2,\zeta} - P^{-2}K_{1,\zeta}$$

then shows that K_2 is a constant.

Theorem 3N: A Type $\mathfrak{A}(2,N)$ space is quasi-diagonal iff it is Type D.

Proof: When a quasi-diagonalizing transformation exists, it may be either calculated formally, using eq. (5.22), or deduced from the metric of eq. (6.8). We shall follow the latter course.

If the metric is q.d., there are no cross-terms between $(dr, d\theta)$ and $(d\eta_1, d\eta_2)$ after a change of ignorable coordinates, $(s, \phi) \to (\eta_1, \eta_2)$. The only such cross-term is eliminated when

$$d\phi + (2\dot{\delta}P^2/\Sigma)(ds + \Lambda d\phi) = d\eta_2 + (2\dot{\delta}P^2/\Sigma)(d\eta_1 + \Lambda d\eta_2) ,$$

$$(W/\Sigma)(ds + \Lambda d\phi) = (W/\Sigma)(d\eta_1 + \Lambda d\eta_2) - dr , \qquad (6.14)$$

or

$$d\eta_1 = ds + [(\Sigma + 2\Lambda\dot{\delta}P^2)/\Sigma] \; dr ,$$

$$d\eta_2 = d\phi - [2\dot{\delta}P^2/W] \; dr . \qquad (6.15)$$

Of course the methods of Chapter V also give eq. (6.15). These equations are integrable iff the coefficients of dr are functions of r; and so the ratio $[P^2\dot{\delta}/(r^2 + \delta^2 + 2\Lambda P^2\dot{\delta})]$ must be independent of θ; i.e. $P^2\dot{\delta}$ and $(\delta^2 + 2\Lambda P^2\dot{\delta})$ must be constants. From eq. (6.10), $P^2\dot{\Lambda} + \delta = 0$, which implies that $(\delta^2 + 2\Lambda\dot{\delta}P^2)_\theta = \Lambda(2\dot{\delta}P^2)_\theta$, and so the ratio is a function of r iff $(P^2\dot{\delta})_\theta$ is zero. But then eq. (6.15) is integrable iff W is a function of r, that is if $(P^2\dot{\delta})_\theta$ is zero and K_1, K_2 and m are constants. This statement, together with Lemma 2N and Lemma 4N, proves Theorem 3N.

The $\mathcal{D}(2,N)$ metrics may now be written as

$$-ds^2 = (\Sigma/2P^2)[d\theta^2 + \{d\eta_2 + (2\dot{\delta}P^2/\Sigma)(d\eta_1 + \Lambda d\eta_2)\}^2]$$

$$+ (\Sigma/W)dr^2 - (W/\Sigma)(d\eta_1 + \Lambda d\eta_2)^2 . \qquad (6.16)$$

Provided that δ is not a constant, we may set

$$\Lambda = -\delta^2/2P^2\dot{\delta} . \qquad (6.17)$$

Under the c_0 transformation of eq. (4.51), $(2P^2\dot{\delta})' = c_0^3(2P^2\dot{\delta})$, and so we shall assume

$$2P^2\dot{\delta} = 1\,, \quad \Lambda = -\delta^2\,. \tag{6.18}$$

If

$$(x,y) = (\delta(\theta), r) \tag{6.19}$$

are to be used as non-ignorable coordinates, then we must know P^2 as a function of x. Recalling eq. (6.18), we find, after defining $V = 1/2P^2$, that

$$\begin{aligned} V = 1/2P^2 &= (2P^2\dot{\delta})^2/2P^2 = 2P^2(\dot{\delta})^2\,, \\ &= 2(\dot{Q})^2 - 2\dot{Q}\dot{P} + (\dot{P})^2\delta^2 \\ &\quad - 2\delta[-\mathrm{Im}(m) + P\ddot{Q} - 2\dot{P}\dot{Q} + \ddot{P}Q]\,, \\ &= 2(\dot{Q}^2 - Q\ddot{Q}) + 2\mathrm{Im}(m)\,\delta + 2\delta^2((\dot{P})^2 - P\ddot{P})\,, \\ &= -2K_2 + 2\mathrm{Im}(m)\,\delta - 2K_1\delta^2\,, \end{aligned} \tag{6.20}$$

where we have used eq. (6.9b) and eq. (4.6). The metric of eq. (6.16) then becomes

$$\begin{aligned} -ds^2 &= (\Sigma/V)\,dx^2 + (V/\Sigma)(d\eta_1 + y^2 d\eta_2)^2 \\ &\quad + (\Sigma/W)\,dy^2 - (W/\Sigma)(d\eta_1 - x^2 d\eta_2)^2\,, \end{aligned} \tag{6.21}$$

where

$$\begin{aligned} V &= -2(K_1x^2 - \mathrm{Im}(m)\,x + K_2)\,, \\ W &= -2(-K_1y^2 + \mathrm{Re}(m)\,y + K_2)\,, \\ \Sigma &= x^2 + y^2\,, \\ \dot{\delta} &\neq 0\,. \end{aligned} \tag{6.22}$$

The curvature invariant, ψ_2, is

$$\psi_2 = - m(y + ix)^{-3} , \tag{6.23}$$

and so the curvature tensor is regular for all points other than $y = 0 = x$. The transformation

$$(x', y', \eta_1', \eta_2', K_1', K_2', m') \rightarrow (qx, qy, q^{-1}\eta_1, q^{-3}\eta_2, q^2K_1, q^4K_2, q^3m) \tag{6.24}$$

preserves the form of eq. (6.21), but modifies the parameters, showing that they are not invariants. If $2K_1 \neq 0$, then it can be chosen to be ± 1. If $V(= 1/2P^2)$ and W are chosen to remain positive, $-K_2$ must be also, whenever $K_1 > 0$ and $\mathrm{Im}(m) = 0$. This leads to the Kerr metric, $V = a - x^2$, $W = y^2 - 2my + a^2$.

When $\dot{\delta} = 0$, then $Q = \delta P$, $K_2 = \delta^2 K_1$ and $\mathrm{Im}(m) = 2\delta K_1$. It is convenient to introduce a new coordinate, x, defined by $(dx/d\theta) = (1/2P^2) = V$ (say), and to choose the s in eq. (4.51) so that $\Lambda = 2\delta x$. It follows from the equation for K_1, eq. (4.6), that

$$\frac{d^2V}{dx^2} + 4K_1 = 0 . \tag{6.25}$$

The metric can then be written as

$$- ds^2 = \Sigma [dx^2/V + dy^2/W + V d\eta_2^2] - (W/\Sigma)(d\eta_1 + 2\delta x d\eta_2)^2 , \tag{6.26}$$

where

$$V = - 2K_1 x^2 + 2qx + p , \tag{6.27a}$$

$$W = 2K_1 (y^2 - \delta^2) - 2m_0 y , \tag{6.27b}$$

$$\Sigma = x^2 + \delta^2 , \tag{6.27c}$$

$$m = m_0 + 2iK_1\delta , \tag{6.27d}$$

$$(y,\eta_1,\eta_2) = (r,s,\phi) , \tag{6.27e}$$

$$\dot{\delta} = 0 , \tag{6.27f}$$

and p and q are arbitrary real constants. This is the B[+] metric[23] of Carter, or the generalized N.U.T. metric.[24] If $K_1 < 0$, then p and q can both be transformed to zero, since if we assume p and q to be zero, then the equation $d\theta = -dx/2K_1x^2$ can be integrated and P deduced to be

$$P = \pm \left(-K_1\right)^{\frac{1}{2}} i(\zeta - \bar{\zeta}) , \tag{6.28}$$

so that K_1 must be negative. On the other hand, if $K_1 < 0$, then we may use the transformation of eq.(4.54) to satisfy eq.(6.28), thereby setting p and q to zero. If $K_1 > 0$, the best that can be done is to eliminate one or other of p or q by the linear transformation $x \to x + x_0$, where x_0 is a real constant. Also, the transformation of eq.(6.24) can be used to set any non-zero $2|K_1|$ to 1, and so the canonical forms for V and W are

$$\begin{aligned} K_1 > 0 \ &: \ V = 1 - x^2 , \quad W = y^2 - 2my - \delta^2 , \\ K_1 < 0 \ &: \ V = x^2 , \quad W = -y^2 - 2my + \delta^2 , \\ K_1 = 0 \ &: \ V = 1 , \quad W = -2my . \end{aligned} \tag{6.29}$$

The corresponding range of coordinates follow from the requirement that V be positive.

The curvature invariant, ψ_2, is given by

$$\psi_2 = -m(y + i\delta)^{-3} , \tag{6.30}$$

and so the curvature tensor is nowhere singular for $\delta \neq 0$. When $\delta = 0$, we get the usual variants of the Schwarzschild metric, which are singular when $y = 0$.

CHAPTER VII

SECTION B

AN EINSTEIN-MAXWELL FIELD WITH A COSMOLOGICAL CONSTANT

In this section we shall generalize the empty space metrics of Section A. We begin by finding charged Type D metrics, with a cosmological constant, which reduce to the metrics of the previous section when the Einstein tensor is set to zero. Conformal Killing tensors are then introduced, and finally a canonical form containing the metrics which were found in the earlier chapters is derived in Chapter 9.

Our approach in seeking charged generalizations to the empty space Type D metrics will be to postulate a canonical form for the metric. We shall then assume that the gravitational field and the Maxwell field for this metric are Type D and non-null respectively. In addition, the two Debever vectors are chosen to be geodesic and shear-free, and to allign with the principal directions of the Maxwell field. We shall orientate our spin basis so that

$$\psi_0 = \psi_1 = \psi_3 = \psi_4 = \phi_0 = \phi_2 = \sigma = \varepsilon = \kappa = \lambda = 0. \quad \psi_2 \neq 0, \phi_1 \neq 0. \qquad (7.1)$$

The conditions $\psi_0 = \psi_1 = \psi_3 = \psi_4 = \phi_0 = \phi_2 = 0$ have been shown by Kinnersley to imply that $\sigma = \varepsilon = \kappa = \lambda = 0$, but apparently there is no theorem which states that $\psi_0 = \psi_1 = \psi_3 = \psi_4 = \sigma = \varepsilon = \kappa = \lambda = 0$ implies $\phi_0 = \phi_2 = 0$.

We shall now explain our choice for the canonical form of the metric. It is known from the work of Kinnersley that all Einstein-

Maxwell fields satisfying eq.(7.1), and with a zero cosmological constant, may be constructed by letting the integration constants in the empty space Type D metrics become explicit functions of the non-ignorable coordinates. The Type C and Kinnersley metrics may therefore be charged by letting the functions g(x),g(y) and $\mathcal{G}(x)$,h(z) in eq.(5.30) and eq.(5.62) become some explicit functions of the non-ignorable coordinates. Furthermore, Carter[23] has shown that this procedure also works for the generalized Kerr-NUT metric which satisfies eq.(7.1) and has a non-zero cosmological constant. When seeking a charged generalization with a cosmological constant to the Type C metric, it is therefore natural to inspect the canonical form

$$ds^2 = -2(x-y)^{-2}[h d\eta_1^2 + dx^2/h + g d\eta_2^2 - dy^2/g] \ , \tag{7.2}$$

where g and h are arbitrary functions of x and y.

The equations we must satisfy are eq.(2.13), eq.(2.16) and eq.(7.1). The term Λ in eq.(2.10) is a real constant, since

$$\partial_{B\dot{B}}\Lambda = \nabla^{A\dot{A}}\,\Phi_{AB\dot{A}\dot{B}} = \nabla^{A\dot{A}}\,\phi_{AB}\,\phi_{\dot{A}\dot{B}} = 0 \ , \tag{7.3}$$

where eq.(2.16) and eq.(2.23b) have been used. From eq.(2.11d), Λ corresponds to the cosmological constant.

The four null vectors

$$\begin{pmatrix}\omega^{0\dot{0}}\\ \omega^{1\dot{1}}\end{pmatrix} = [dy/g^{\frac{1}{2}} \mp g^{\frac{1}{2}}d\eta_2]\,/\,(x-y) \ , \tag{7.4a}$$

$$\begin{pmatrix}\omega^{0\dot{1}}\\ \omega^{1\dot{0}}\end{pmatrix} = [dx/h^{\frac{1}{2}} \pm ih^{\frac{1}{2}}d\eta_1]\,/\,(x-y) \ , \tag{7.4b}$$

satisfy the relationship $ds^2 = 2(\omega^{0\dot{0}}\,\omega^{1\dot{1}} - \omega^{0\dot{1}}\,\omega^{1\dot{0}})$, and so will be chosen as our spin basis for one-forms.

A straightforward calculation yields

$$\kappa = h^{\frac{1}{2}} g_x / 4g \qquad , \tag{7.5a}$$

$$\sigma = (x-y)\, g^{\frac{1}{2}} h_y / 4h \, , \tag{7.5b}$$

and so, from eq.(7.1), we must have

$$g = g(y) \; , \quad h = h(x) \; . \tag{7.6}$$

Imposing these conditions gives the following spin coefficients,

$$\alpha = -\beta = h^{\frac{1}{2}}/4 - (x-y)h_x / 8h^{\frac{1}{2}} \, , \tag{7.7a}$$

$$\tau = -\pi = \tfrac{1}{2}h^{\frac{1}{2}} \qquad , \tag{7.7b}$$

$$\rho = -\mu = \tfrac{1}{2}g^{\frac{1}{2}} \qquad , \tag{7.7c}$$

$$\varepsilon = -\gamma = g^{\frac{1}{2}}/4 + (x-y)g_y / 8g^{\frac{1}{2}} \, , \tag{7.7d}$$

$$\nu = \sigma = \lambda = \kappa = 0 \qquad , \tag{7.7e}$$

from which we compute

$$4\Theta_{00} = [\,2h - 2g - (x-y)\ (h_x + g_y)]\ \ \omega^{1\dot{1}} \wedge \omega^{1\dot{0}} \, , \tag{7.8a}$$

$$4\Theta_{11} = [\,2h - 2g - (x-y)\ (h_x + g_y)]\ \ \omega^{0\dot{1}} \wedge \omega^{0\dot{0}} \, , \tag{7.8b}$$

$$\begin{aligned} 8\Theta_{01} = {} & [\,2h - 2g - 2(x-y)\, g_y - (x-y)^2\, g_{yy}]\ \ \omega^{0\dot{0}} \wedge \omega^{1\dot{1}} \\ & + [\,2h - 2g - 2(x-y)h_x + (x-y)^2\, h_{xx}]\ \ \omega^{1\dot{0}} \wedge \omega^{0\dot{1}} \, , \end{aligned} \tag{7.8c}$$

where

$$\begin{aligned} 2\Theta_{AB} &= 2d\Gamma_{AB} + 2\Gamma_{AC} \wedge \Gamma^{C}{}_{B} \qquad , \\ &= \psi_{ABCD}\, \omega^{C}{}_{\dot{D}} \wedge \omega^{D\dot{D}} + \Phi_{AB\dot{C}\dot{D}}\, \omega_{D}{}^{\dot{C}} \wedge \omega^{D\dot{D}} \\ &\quad - 2\Lambda\, \epsilon_{AC}\, \epsilon_{BD}\, \omega^{C}{}_{\dot{D}} \wedge \omega^{D\dot{D}} \, . \end{aligned} \tag{7.9}$$

Eq. (7.9) is simply a restatement of eq. (2.13).

The source-free Maxwell equations, eqs. (2.16), may be written as

$$0 = \epsilon^{AD} \nabla_{D\dot{E}} \phi_{AB} = \epsilon^{AD}[\partial_{D\dot{E}} \phi_{AB} - \Gamma^{F}{}_{AD\dot{E}} \phi_{FB} - \Gamma^{F}{}_{BD\dot{E}} \phi_{AF}] .$$

Substituting the spin coefficients in eq. (7.7) into this equation, and using the notation of eq. (2.18), we deduce that

$$\partial_{0\dot{0}} \phi_1 = -g^{\frac{1}{2}} \phi_1 , \tag{7.10a}$$

$$\partial_{0\dot{1}} \phi_1 = h^{\frac{1}{2}} \phi_1 , \tag{7.10b}$$

$$\partial_{1\dot{0}} \phi_1 = h^{\frac{1}{2}} \phi_1 , \tag{7.10c}$$

$$\partial_{1\dot{1}} \phi_1 = -g^{\frac{1}{2}} \phi_1 . \tag{7.10d}$$

These equations can be solved by expressing the spin derivatives, $\partial_{A\dot{A}}$, in terms of holonomic derivatives. Explicitly, we write $\partial_{A\dot{A}} = e_{A\dot{A}}{}^{\mu} \partial_{\mu}$, where $e_{A\dot{A}}{}^{\mu} e^{B\dot{B}}{}_{\mu} = \delta^{A}_{B} \delta^{\dot{A}}_{\dot{B}}$. The $e^{B\dot{B}}{}_{\mu}$ are given in eq. (7.4). We find that

$$\partial_{0\dot{0}} = \tfrac{1}{2}(x-y)[g^{\frac{1}{2}} \partial_y - g^{-\frac{1}{2}} \partial_{\eta_2}] , \tag{7.11a}$$

$$\partial_{1\dot{1}} = \tfrac{1}{2}(x-y)[g^{\frac{1}{2}} \partial_y + g^{-\frac{1}{2}} \partial_{\eta_2}] , \tag{7.11b}$$

$$\partial_{0\dot{1}} = \tfrac{1}{2}(x-y)[h^{\frac{1}{2}} \partial_x - ih^{-\frac{1}{2}} \partial_{\eta_1}] , \tag{7.11c}$$

$$\partial_{1\dot{0}} = \overline{\partial_{0\dot{1}}} . \tag{7.11d}$$

A straightforward calculation gives as the general solution to eq. (7.10),

$$\phi_1 = a(x-y)^2 , \tag{7.12}$$

where a is a complex constant. The vector potential, A, associated with ϕ_1 is defined, up to an arbitrary closed one-form, by the equation

$$2dA = [\epsilon_{AB}\,\phi_{\dot{A}\dot{B}} + \epsilon_{\dot{A}\dot{B}}\,\phi_{AB}]\,\omega^{A\dot{A}}\wedge\omega^{B\dot{B}} \qquad ,$$

$$= 2(a+\bar{a})\ dy\wedge d\eta_2 + 2i(a-\bar{a})\ dx\wedge d\eta_1\ , \qquad (7.13)$$

from which we conclude that

$$A = 2(a+\bar{a})y\,d\eta_2 + 2i\ (a-\bar{a})x\,d\eta_1 + d\psi\ , \qquad (7.14)$$

where ψ is an arbitrary real function of the four coordinates x, y, η_1 and η_2.

The remaining equations to be solved are eq. (2.16c) and eq. (7.9). Forming $\partial_x\partial_x\partial_y\partial_y[\Theta_{00}-\Theta_{11}]$ from eq. (7.9) yields

$$h_{xxxx} + g_{yyyy} + 192\ |a|^2 = 0\ . \qquad (7.15)$$

We see that h and g must both be quartic functions, and so eq. (7.9) gives

$$g = -4\ |a|^2\,y^4 + b_1\,y^3 + b_2\,y^2 + b_3 y + b_4 + 2\Lambda\ , \qquad (7.16a)$$

$$h = -4\ |a|^2\,x^4 + b_1\,x^3 + b_2\,x^2 + b_3 x + b_4 - 2\Lambda\ , \qquad (7.16b)$$

$$\phi_1 = a(x-y)^2 \qquad , \qquad (7.16c)$$

$$\psi_2 = (x-y)^3\left[\frac{b_1}{4} - 2\ |a|^2\ (x+y)\right] \qquad , \qquad (7.16d)$$

where b_1, b_2, b_3 and b_4 are real constants. This completes the solution for the metric given in eq. (7.2), and satisfying eq. (2.16c) and eq. (7.1).[25]

From these results, it is natural to inspect the canonical form

$$ds^2 = -2(1-xy)^{-2}\,[\,(\Sigma/h)\,dx^2 + (h/\Sigma)\,(d\eta_2 + y^2\,d\eta_1)^2 + (\Sigma/g)\,dy^2$$

$$- (g/\Sigma)\,(d\eta_2 - x^2\,d\eta_1)^2]\ , \tag{7.17}$$

$$\Sigma = x^2 + y^2\ , \tag{7.18}$$

when seeking a generalization for the Kinnersley metric. Here h and g are functions of x and y respectively.

Our spin basis shall be chosen such that

$$\begin{pmatrix}\omega^{0\dot{0}}\\ \omega^{1\dot{1}}\end{pmatrix} = [\,(g/\Sigma)^{\frac{1}{2}}\ (d\eta_2 - x^2\,d\eta_1)\ \pm\ (\Sigma/g)^{\frac{1}{2}}\,dy]\ /\ (1-xy)\ , \tag{7.19a}$$

$$\begin{pmatrix}\omega^{0\dot{1}}\\ \omega^{1\dot{0}}\end{pmatrix} = [\,(\Sigma/h)^{\frac{1}{2}}\,dx\ \pm\ i\,(h/\Sigma)^{\frac{1}{2}}\,(d\eta_2 + y^2\,d\eta_1)]\,/\,(1-xy)\ . \tag{7.19b}$$

This leads to the following spin coefficients,

$$\rho = \mu = -\,xg^{\frac{1}{2}}/2\Sigma^{\frac{1}{2}} - (1-xy)\,g^{\frac{1}{2}}\,(y - ix)/2\Sigma^{3/2}\ ,$$

$$\varepsilon = \gamma = xg^{\frac{1}{2}}/4\ ^{\frac{1}{2}} + \tfrac{1}{4}(1-xy)[\,(g/\Sigma)^{\frac{1}{2}}]_y\ +\ i\,(1-xy)\ xg^{\frac{1}{2}}/4\Sigma^{3/2}\ ,$$

$$\beta = -\,\alpha = yg^{\frac{1}{2}}/4\Sigma^{\frac{1}{2}} + \tfrac{1}{4}(1-xy)[\,(h/\Sigma)^{\frac{1}{2}}]_y\ -\ i\,(1-xy)\ yh^{\frac{1}{2}}/4\Sigma^{3/2}\ ,$$

$$\pi = \tau = yh^{\frac{1}{2}}/2\Sigma^{\frac{1}{2}} + (1-xy)\ h^{\frac{1}{2}}\ (x + iy)/2\Sigma^{3/2}\ ,$$

$$\kappa = \sigma = \lambda = \nu = 0\ , \tag{7.20}$$

from which we calculate

$$\Theta_{00} = [\,(xy-1)\,(y-ix)\,(1-ix^2)\,g_y\,/\,4\Sigma^2 + (xy-1)\,(x+iy)\,(1-iy^2)\,h_x\,/\,4\Sigma^2$$

$$+\ g\{-\,x^2/2\Sigma + (1-xy)^2\,y\,(y-ix)/\Sigma^3 - (1-xy)^2/2\Sigma^2\}$$

$$+\ h\{-\,y^2/2\Sigma + (1-xy)^2\,x\,(x+iy)/\Sigma^3 - (1-xy)^2/2\Sigma^2\}]\,\omega^{1\dot{0}}\wedge\,\omega^{1\dot{1}}\ ,$$

$$= \Theta_{001\dot{0}\,1\dot{1}}\ \omega^{1\dot{0}}\wedge\ \omega^{1\dot{1}}\ ,$$

$$\Theta_{11} = \Theta_{110\dot{0}0\dot{1}}\, \omega^{0\dot{0}} \wedge \omega^{0\dot{1}} ,$$

$$\Theta_{110\dot{0}0\dot{1}} = \Theta_{001\dot{0}1\dot{1}} ,$$

$$\Theta_{01} = \Theta_{010\dot{0}1\dot{1}}\, \omega^{0\dot{0}} \wedge \omega^{1\dot{1}} + \Theta_{010\dot{1}1\dot{0}}\, \omega^{0\dot{1}} \wedge \omega^{1\dot{0}} ,$$

$$\begin{aligned}
\Theta_{010\dot{0}1\dot{1}} = {} & (1-xy)^2 g_{yy}/8\Sigma + g_y[(1-xy)\, x/4\Sigma \\
& + i(1-xy)^2 x/4\Sigma^2 - (1-xy)^2 y/2\Sigma^2] \\
& + g[x^2/4\Sigma - (1-xy)^2/4\Sigma^2 + (1-xy)^2 y^2/\Sigma^3 \\
& - (1-xy)\, xy/2\Sigma^2 - i(1-xy)^2 xy/\Sigma^3] \\
& + h_x[-iy(1-xy)^2/4\Sigma^2] + h[y^2/4\Sigma + (1-xy)\, xy/2\Sigma^2 \\
& + i(1-xy)^2 xy/\Sigma^3 + (1-xy)^2 x^2/4\Sigma^3 - 3(1-xy)^2 y^2/4\Sigma^3] ,
\end{aligned}$$

$$\begin{aligned}
\Theta_{010\dot{1}1\dot{0}} = {} & -(1-xy)^2 h_{xx}/8\Sigma + h_x[x(1-xy)^2/2\Sigma^2 \\
& + i(1-xy)^2 y/4\Sigma^2 - (1-xy)\, y/4\Sigma] \\
& + h[(1-xy)^2/4\Sigma^2 - x^2(1-xy)^2/\Sigma^3 - i(1-xy)^2 xy/\Sigma^3 \\
& - y^2/4\Sigma + xy(1-xy)/2\Sigma^2] + g_y[-i(1-xy)^2 x/4\Sigma^2] \\
& + g[-x^2/4\Sigma - (1-xy)^2 y^2/4\Sigma^3 + 3(1-xy)^2 x^2/4\Sigma^3 \\
& - (1-xy)\, xy/2\Sigma^2 + i(1-xy)^2 xy/\Sigma^3] . \qquad (7.21)
\end{aligned}$$

The two-forms, Θ_{AB}, are defined in eq.(7.9), and the quantity $\Theta_{ABC\dot{C}D\dot{D}}$ is the coefficient of $\omega^{C\dot{C}} \wedge \omega^{D\dot{D}}$ in Θ_{AB}.

The source-free Maxwell equations, eq.(2.16), subject to eq.(7.1), are

$$\partial_{0\dot{0}}\, \phi_1 = 2\rho\, \phi_1 , \qquad (7.22a)$$

$$\partial_{0\dot{1}}\,\phi_1 = 2\tau\,\phi_1 \quad , \tag{7.22b}$$

$$\partial_{1\dot{0}}\,\phi_1 = -2\pi\,\phi_1 \quad , \tag{7.22c}$$

$$\partial_{1\dot{1}}\,\phi_1 = -2\mu\,\phi_1 \quad , \tag{7.22d}$$

where

$$\partial_{0\dot{0}} = (1-xy)\,[g\partial_y + y^2\partial_{\eta_2} - \partial_{\eta_1}]/2\Sigma^{\frac{1}{2}} g^{\frac{1}{2}} \quad , \tag{7.23a}$$

$$\partial_{1\dot{1}} = (1-xy)\,[-g\partial_y + y^2\,\partial_{\eta_2} - \partial_{\eta_1}]/2\; g^{\frac{1}{2}}\Sigma^{\frac{1}{2}} \; , \tag{7.23b}$$

$$\partial_{0\dot{1}} = (1-xy)\,[h\partial_x - ix^2\,\partial_{\eta_2} - i\partial_{\eta_1}]/2\,h^{\frac{1}{2}}\Sigma^{\frac{1}{2}} \; , \tag{7.23c}$$

$$\partial_{1\dot{0}} = \overline{\partial_{0\dot{1}}} \quad , \tag{7.23d}$$

and the spin coefficients are given in eq. (7.20). The general solution for ϕ_1 is

$$\phi_1 = a[\,1 - xy/(y+ix)\,]^2 \; , \tag{7.24}$$

where a is a complex constant. The corresponding one-form, A, defined in eq. (7.13a), is

$$A = 2\bar{a}\,(ixyd\eta_1 - d\eta_2)/(y-ix) \;-\; 2a\,(d\eta_2 + ixyd\eta_1)/(y+ix) \;+\; d\psi \; , \tag{7.25}$$

where ψ is an arbitrary function of the coordinates x, y, η_1 and η_2.

The remaining equations may now be solved. Forming $\partial_x\partial_x\partial_y\partial_y[\Theta_{001\dot{0}1\dot{1}} + \Theta_{110\dot{0}0\dot{1}}]$ from eq. (7.21) yields

$$\partial_x[\,xh_{xxxx}] \;-\; \partial_y[\,yg_{yyyy}] \;+\; 96\,|a|^2 = 0 \, . \tag{7.26}$$

Integrating this expression, and using the remaining equations,

$$h = (b_1 - 2|a|^2 + 2\Lambda)x^4 + b_2x^3 + b_3x^2 + b_4x + 2\Lambda - b_1 - 2|a|^2 ,$$

$$g = (b_1 + 2|a|^2 + 2\Lambda)y^4 - b_4y^3 - b_3y^2 - b_2y + 2\Lambda - b_1 + 2|a|^2 ,$$

$$\phi_1 = a[(1-xy)/(y+ix)]^2 ,$$

$$\psi_2 = [(1-xy)/(x-iy)]^3 [(b_4 - ib_2)/4 - 2|a|^2(1+xy) /(x+iy)] , \tag{7.27}$$

where b_1, b_2, b_3 and b_4 are real constants. This completes the solution for the metric given in eq.(7.17), and satisfying eq.(2.17c) and eq.(7.1). It is a metric found recently by J. Plebanski.[26]

This metric has two important limits. The first is found by defining $b_1' = b_1 - 2\Lambda$, and dropping the prime. We then apply the transformation

$$(x, y, \eta_1, \eta_2, |a|, b_1, b_2, b_3, b_4, \Lambda) \rightarrow$$

$$(\alpha x, \alpha y, \alpha^{-3}\eta_1, \alpha^{-1}\eta_2, \alpha^2|a|, \alpha^4 b_1, \alpha^3 b_2, \alpha^2 b_3, \alpha^3 b_4, \Lambda) , \tag{7.28}$$

and set $\alpha = 0$. The resultant metric is

$$-\tfrac{1}{2}ds^2 = \frac{\Sigma dx^2}{h} + \frac{h}{\Sigma}(d\eta_2 + y^2 d\eta_1)^2 + \frac{\Sigma dy^2}{g} - \frac{g}{\Sigma}(d\eta_2 - x^2 d\eta_1)^2 , \tag{7.29}$$

where

$$\Sigma = x^2 + y^2 , \tag{7.30a}$$

$$h = 4\Lambda x^4 + b_3x^2 + b_4x - b_1 - 2|a|^2 , \tag{7.30b}$$

$$g = 4\Lambda y^4 - b_3y^2 - b_2y - b_1 + 2|a|^2 . \tag{7.30c}$$

This is the generalized Kerr-NUT metric found by Carter.[23]

A second limit of eq.(7.27) is found by defining $b_1' = b_1 - 2|a|^2$, and dropping the prime. We then apply the transformation

$$(x, y, \eta_1, \eta_2, |a|, b_1, b_2, b_3, b_4, \Lambda) \rightarrow$$

$$(\alpha^{-1}x, \alpha y, \alpha\eta_1, \alpha\eta_2, \alpha^{-2}|a|, b_1, \alpha^{-1}b_2, \alpha^{-2}b_3, \alpha^{-3}b_4, \Lambda) \ , \qquad (7.31)$$

and set $\alpha = 0$. The resultant metric is

$$ds^2 = \frac{-2}{(x-y)^2}\left[\frac{dx^2}{h} + h d\eta_2^2 - \frac{dy^2}{g} + g d\eta_1^2\right] , \qquad (7.32)$$

where

$$g = -4|a|^2 y^4 + b_4 y^3 + b_3 y^2 + b_2 y + b_1 - 2\Lambda ,$$

$$h = -4|a|^2 x^4 + b_4 x^3 + b_3 x^2 + b_2 x + b_1 + 2\Lambda . \qquad (7.33)$$

This is the generalized C metric of eq. (7.16).

CHAPTER VIII

CONFORMAL KILLING TENSORS

Some knowledge of the geodesics in an Einstein space is necessary both for a discussion of test sources of electromagnetic radiation and uncharged matter,[27] and for investigating geodesic completeness.[28] To solve the geodesic equations, however, requires four first integrals. The remarkable fact that these integrals may be found explicitly in all empty non-radiating Type D metrics was first found by Carter.[11] This result prompted further research, and in a paper by Walker and Penrose[10] it was proved that all empty Type D spaces possess a non-redundant conformal Killing tensor. C.K.T.'s have also been discussed by R. Geroch,[9] P. Sommers[29] and N.M.J. Woodhouse.[13]

In this section we shall investigate C.K.T.'s, closely following the work of Hauser and Malhiot[12] on Killing tensors.

Definition: A symmetric tensor, $Q^{(\alpha..\beta\gamma)}$, with r contravariant indices, $\alpha..\beta\gamma$, is a conformal Killing tensor of order r iff

$$Q^{(\alpha..\beta\gamma;\delta)} = H^{(\alpha..\beta}g^{\gamma\delta)}\ , \tag{8.1}$$

where $H^{(\alpha..\beta)}$ is some symmetric tensor with $(r-1)$ contravariant indices, and $g^{\gamma\delta}$ is the metric tensor. If $H^{(\alpha..\beta)}$ is zero, $Q^{(\alpha..\beta\gamma)}$ is called a Killing tensor.

For the remainder of this thesis we shall discuss only second order C.K.T.'s, $Q^{(\alpha\beta)}$, and any reference to a C.K.T. will refer to this subclass. We shall work in an n dimensional space for this chapter, so that

$$g^{\alpha}{}_{\alpha} = n\,. \tag{8.2}$$

It follows from eq.(8.1) that any symmetrized product of conformal Killing vectors is a C.K.T., as is the metric tensor.

Definition: A conformal Killing tensor is called redundant iff it is a linear sum, with constant coefficients, of a multiple of the metric tensor and symmetrized products of conformal Killing vectors[30]. Otherwise it is called non-redundant.

The main difference between a C.K.T. and a Killing tensor is that only the trace-free part of a C.K.T. need be considered. This follows largely from the following well-known lemma,

Lemma K1: Given any (null) geodesic, ξ^{α}, and some symmetric tensor, $Q_{\alpha\beta}$, the scalar $Q_{\alpha\beta}\,\xi^{\alpha}\xi^{\beta}$ is constant along this (null) geodesic iff $Q_{\alpha\beta}$ is a (conformal) Killing tensor.

Proof: Let s be an affine parameter along ξ^{α}. Then

$$\begin{aligned}\frac{d}{ds}\,(Q_{\alpha\beta}\,\xi^{\alpha}\xi^{\beta}) &= Q_{(\alpha\beta;\gamma)}\,\xi^{\alpha}\xi^{\beta}\xi^{\gamma} + 2Q_{\alpha\beta}\,\xi^{\alpha}\xi^{\beta}{}_{;\gamma}\,\xi^{\gamma}\,,\\ &= Q_{(\alpha\beta;\gamma)}\,\xi^{\alpha}\xi^{\beta}\xi^{\gamma}\,,\end{aligned} \tag{8.3}$$

since ξ^{α} is geodesic. But the right hand side of eq.(8.3) is zero iff eq.(8.1) holds.

C.K.T.'s are of interest because they define quadratic constants of the motion along a null geodesic. However, only the trace-free part of a C.K.T. will contribute to this constant. In particular, if

$$\hat{Q}_{\alpha\beta} = Q_{\alpha\beta} - Q^{\gamma}{}_{\gamma}\,g_{\alpha\beta}/n\,, \tag{8.4a}$$

then for any null geodesic, ξ^{α},

$$\hat{Q}_{\alpha\beta}\,\xi^{\alpha}\xi^{\beta} = Q_{\alpha\beta}\,\xi^{\alpha}\xi^{\beta}\,. \tag{8.4b}$$

Furthermore, $\hat{Q}_{\alpha\beta}$ is a C.K.T., and

$$\hat{Q}_{(\alpha\beta;\gamma)} = \hat{H}_{(\alpha}g_{\beta\gamma)}\,, \tag{8.4c}$$

$$\hat{Q}^{\alpha}{}_{\alpha} = 0\,, \tag{8.4d}$$

$$(n+2)\,\hat{H}_{\alpha} = 2\,Q^{\beta}{}_{\alpha;\beta}\,. \tag{8.4e}$$

An arbitrary Riemannian space will not, in general, contain a non-redundant C.K.T., but given any metric, the dimension of the linear vector space of C.K.T.'s formed from sums of C.K.T.'s with real coefficients is bounded. Finding this upper bound will occupy the rest of this chapter. First, we shall define

$$L_{\alpha\beta\gamma} = Q_{\beta\gamma;\alpha} - Q_{\alpha\gamma;\beta}\,, \tag{8.5}$$

$$M_{\alpha\beta\gamma\delta} = \tfrac{1}{2}(L_{\alpha\beta[\gamma;\delta]} + L_{\gamma\delta[\alpha;\beta]})\,, \tag{8.6}$$

$$H_{\alpha\beta} = H_{[\alpha;\beta]}\,, \tag{8.7}$$

$$H_{\alpha\beta\gamma} = H_{[\alpha;\beta];\gamma}\,, \tag{8.8}$$

$$I_{\alpha\beta} = H^{\gamma}{}_{;\gamma;(\alpha;\beta)}\,, \tag{8.9}$$

$$2\Theta_{\alpha\beta\gamma} = 3H_{(\alpha}g_{\beta\gamma)}\,, \tag{8.10}$$

$$M_{\alpha\beta} = g^{\gamma\delta}M_{\alpha\gamma\beta\delta}\,, \tag{8.11}$$

$$M = g^{\alpha\beta}M_{\alpha\beta}\,, \tag{8.12}$$

where $Q_{(\alpha\beta;\gamma)} = H_{(\alpha}g_{\beta\gamma)}$. We then impose the further condition that

$$Q^{\alpha}{}_{\alpha} = 0\,. \tag{8.13}$$

From eq.(8.5) through eq.(8.7),

$$L_{[\alpha\beta\gamma]} = 0\,, \tag{8.14}$$

$$M_{\alpha[\beta\gamma\delta]} = 0 \tag{8.15}$$

$$H_{[\alpha\beta\gamma]} = 0\,. \tag{8.16}$$

Our immediate aim is to express the derivatives of $Q_{\alpha\beta}$ and H_α as a linear sum of the variables defined in eq.(8.5) through eq.(8.9). From eqs.(8.5,1,10)(†),

$$\tfrac{3}{2} Q_{\beta\gamma;\alpha} = L_{\alpha(\beta\gamma)} + \Theta_{\alpha\beta\gamma}\,. \tag{8.17}$$

To calculate the derivatives of $L_{\alpha\beta\gamma}$, we use eqs.(8.5,1,10) and the Riemann identities to find

$$L_{\alpha\beta(\gamma;\delta)} = Q_{\beta\gamma;[\alpha\delta]} + Q_{\alpha\gamma;[\delta\beta]} + Q_{\delta\alpha;[\gamma\beta]}$$

$$+ Q_{\beta\delta;[\alpha\gamma]} + Q_{\gamma\delta;[\alpha\beta]} + 2\Theta_{\gamma\delta[\beta;\alpha]}\,, \tag{8.18a}$$

$$= 2R_{\alpha\beta\theta(\delta}Q_{\gamma)}{}^{\theta} + 2Q^{\theta}{}_{[\alpha}R_{\beta](\gamma\delta)\theta} + 2\Theta_{\gamma\delta[\beta;\alpha]}\,. \tag{8.18b}$$

The Riemann identities also give

$$L_{\alpha\beta[\gamma;\delta]} - L_{\gamma\delta[\alpha;\beta]} = Q_{\beta\gamma;[\alpha\delta]} + Q_{\alpha\gamma;[\delta\beta]}$$

$$+ Q_{\alpha\delta;[\beta\gamma]} + Q_{\beta\delta;[\gamma\alpha]}\,, \tag{8.19a}$$

$$= R_{\alpha\beta\theta[\gamma}Q_{\delta]}{}^{\theta} - R_{\gamma\delta\theta[\alpha}Q_{\beta]}{}^{\theta}\,, \tag{8.19b}$$

and so, from eqs.(8.6,18,19),

(†) This obvious notation means eq.(8.5), eq.(8.1), and eq.(8.10).

$$L_{\alpha\beta\gamma;\delta} = M_{\alpha\beta\gamma\delta} + \tfrac{1}{2}(L_{\alpha\beta[\gamma;\delta]} - L_{\gamma\delta[\alpha;\beta]})$$

$$+ L_{\alpha\beta(\gamma;\delta)} , \tag{8.20a}$$

$$= M_{\alpha\beta\gamma\delta} + \tfrac{1}{2}R_{\alpha\beta\theta[\gamma}Q_{\delta]}{}^{\theta} - \tfrac{1}{2}R_{\gamma\delta\theta[\alpha}Q_{\beta]}{}^{\theta}$$

$$- 2R_{\alpha\beta\theta(\delta}Q_{\gamma)}{}^{\theta} - 2Q^{\theta}{}_{[\alpha}R_{\beta](\gamma\delta)\theta}$$

$$+ 2\Theta_{\gamma\delta[\beta;\alpha]} . \tag{8.20b}$$

Covariantly differentiating eq.(8.18), and using eqs.(8.17,10,14), together with the Bianchi identities,

$$L_{\alpha\beta(\gamma;\mu)\delta} - L_{\alpha\beta(\delta;\mu)\gamma} = 2R_{\alpha\beta\theta\mu;[\gamma}Q_{\delta]}{}^{\theta}$$

$$- R_{\alpha\beta\delta\gamma;\theta}Q_{\mu}{}^{\theta} + Q^{\theta}{}_{[\alpha}R_{\beta]\mu\delta\gamma;\theta} + R_{\mu\theta\gamma\delta;[\beta}Q_{\alpha]}{}^{\theta}$$

$$+ R_{\alpha\beta\mu}{}^{\theta}\,L_{\delta\gamma\theta} + H_{[\alpha}R_{\beta]\mu\delta\gamma} + \tfrac{2}{3}\,R_{\alpha\beta\gamma\delta}\,H_{\mu}$$

$$+ \tfrac{1}{6}\,R_{\alpha\beta\mu[\delta}H_{\gamma]} + \tfrac{1}{6}\,R_{\alpha\beta}{}^{\theta}{}_{[\delta}g_{\gamma]\mu}H_{\theta}$$

$$+ 2g_{\mu[\alpha}R_{\beta][\gamma\delta]}{}^{\theta}\,H_{\theta} + \tfrac{2}{3}\,g_{[\gamma}[{}_{\alpha}R_{\beta}]{}_{\delta]\mu}{}^{\theta}\,H_{\theta}$$

$$+ \tfrac{2}{3}\,g_{[\gamma[\alpha}R_{\beta]|\mu|\delta]}{}^{\theta}\,H_{\theta} + \tfrac{1}{3}\,\delta^{\phi\chi}_{\alpha\beta}\,\delta^{\psi\omega}_{\gamma\delta}\Big[R_{\phi\chi\psi}{}^{\theta}\,L_{\omega(\mu\theta)}$$

$$+ 2R_{\phi(\mu\psi)}{}^{\theta}\,L_{\omega(\theta\chi)} + 3\Theta_{\phi\omega\mu;\chi\psi}\Big] , \tag{8.21a}$$

$$\delta^{\phi\chi}_{\alpha\beta} = \delta^{\phi}_{\alpha}\,\delta^{\chi}_{\beta} - \delta^{\phi}_{\beta}\,\delta^{\chi}_{\alpha} . \tag{8.21b}$$

From the Riemann identities,

$$L_{\alpha\beta(\gamma;\mu)\delta} - L_{\alpha\beta(\delta;\mu)\gamma} = L_{\alpha\beta\gamma;[\mu\delta]} + L_{\alpha\beta\mu;[\gamma\delta]}$$

$$+ L_{\alpha\beta\delta;[\gamma\mu]} + L_{\alpha\beta[\gamma;\delta]\mu} , \tag{8.22a}$$

$$= L_{\alpha\beta[\gamma;\delta]\mu} + R_{\gamma\delta\mu}{}^{\theta}\,L_{\alpha\beta\theta}$$

$$+ \delta^{\phi\chi}_{\alpha\beta}\,\delta^{\psi\omega}_{\gamma\delta}\Big[R_{\mu\psi\phi}{}^{\theta}\,L_{\chi\theta\omega} - \tfrac{1}{2}\,R_{\psi\omega\phi}{}^{\theta}\,L_{\chi\theta\mu}\Big] . \tag{8.22b}$$

Then, from eq.(8.6) and eq.(8.22b),

$$M_{\alpha\beta\gamma\delta;\mu} = \tfrac{1}{2}L_{\alpha\beta[\gamma;\delta];\mu} + \tfrac{1}{2}L_{\gamma\delta[\alpha;\beta];\mu}\,, \tag{8.23a}$$

$$\begin{aligned} &= \tfrac{1}{2}(L_{\alpha\beta(\gamma;\mu);\delta} - L_{\alpha\beta(\delta;\mu);\gamma}) \\ &+ \tfrac{1}{2}(L_{\gamma\delta(\alpha;\mu);\beta} - L_{\gamma\delta(\beta;\mu);\alpha}) \\ &- \tfrac{1}{2}R_{\gamma\delta\mu}{}^{\theta}\, L_{\alpha\beta\theta} - \tfrac{1}{2}R_{\alpha\beta\mu}{}^{\theta}\, L_{\gamma\delta\theta} \\ &+ \tfrac{1}{2}\left[\delta^{\phi\chi}_{\alpha\beta}\,\delta^{\psi\omega}_{\gamma\delta} + \delta^{\phi\chi}_{\gamma\delta}\,\delta^{\psi\omega}_{\alpha\beta}\right]\left[\tfrac{1}{2}R_{\psi\omega\phi}{}^{\theta}\, L_{\chi\theta\mu} - R_{\mu\psi\phi}{}^{\theta}\, L_{\chi\theta\omega}\right]. \end{aligned} \tag{8.23b}$$

By using eq.(8.21) to substitute for the first four terms in eq.(8.23b), we may establish

<u>Lemma K2</u>: The first derivatives of $M_{\alpha\beta\gamma\delta}$ are a linear sum of $Q_{\alpha\beta}$, $L_{\alpha\beta\gamma}$, $M_{\alpha\beta\gamma\delta}$, H_α and $H_{\alpha;\beta;\gamma}$, whose coefficients are concomitants(†) of the metric tensor, the Riemann tensor and its covariant derivatives.

We shall now investigate the derivatives of H_α. From eqs.(8.1,2,13,5),

$$H_\alpha = (2/n+2)\, g^{\beta\gamma}\, L_{\beta\alpha\gamma}\,. \tag{8.24}$$

Forming $g^{\beta\gamma}\, L_{\alpha\beta\gamma;\delta}$ from eq.(8.20), and using eqs.(8.2,11) and the Riemann tensor symmetries, gives

$$(n+1)\, H_{(\alpha;\beta)} = M_{\alpha\beta} + \tfrac{3}{2} R_{\gamma(\beta}Q_{\alpha)}{}^{\gamma} + \tfrac{3}{2} R_{\alpha\gamma\delta\beta}\, Q^{\gamma\delta} + g_{\alpha\beta}\, M/(n+2)\,, \tag{8.25}$$

(†) If a set of quantities is given, every quantity whose components can be expressed as functions of the components of the quantities of the set is called a concomitant of the set[31]. To save space, we shall refer to the concomitants of the members of a set.

$$H^{\alpha}{}_{\alpha} = 2M/(n+2)\ . \tag{8.26}$$

The skew derivative of H_α may be found by inner producting eq. (8.20) by $g^{\gamma\delta}$,

$$(n+2)\, H_{[\alpha;\beta]} = g^{\gamma\delta}\, L_{\beta\alpha(\gamma;\delta)} + 2Q^{\gamma}{}_{[\alpha}R_{\beta]\gamma}\ . \tag{8.27}$$

Before proceeding, some notation is needed. We shall define the following sets,

$$S_0 = \{Q_{\alpha\beta}\}\ ,\ \ S_1 = \{Q_{\alpha\beta;\gamma},\ L_{\alpha\beta\gamma},\ H_\alpha\}\ ,$$

$$S_2 = \{Q_{\alpha\beta;\gamma;\delta},\ L_{\alpha\beta\gamma;\delta},\ M_{\alpha\beta\gamma\delta},\ H_{\alpha;\beta}\}\ ,\ \text{etc;}$$

which allows an ordering defined by $x \leqslant y$ iff $x \in S_a$, $y \in S_b$, and $a \leqslant b$. We shall say that x has order a if $x \in S_a$. It is obvious that this ordering is well-defined, provided we consider only non-zero terms. It can be used to group the lower order, (and lengthy), terms in an expression into a single term. Explicitly, we shall define the tensors, $\hat{L}_{\alpha\beta\gamma\delta}$ and $\hat{M}_{\alpha\beta\gamma\delta\mu}$, through the following equations,

$$L_{\alpha\beta\gamma;\delta} = M_{\alpha\beta\gamma\delta} + 2\,\Theta_{\gamma\delta[\beta;\alpha]} + \hat{L}_{\alpha\beta\gamma\delta}\ , \tag{8.28}$$

$$M_{\alpha\beta\gamma\delta;\mu} = \tfrac{1}{2}\left(\delta^{\phi\psi}_{\alpha\beta}\,\delta^{\omega\chi}_{\gamma\delta} + \delta^{\phi\psi}_{\gamma\delta}\,\delta^{\omega\chi}_{\alpha\beta}\right)\Theta_{\phi\chi\mu;\psi\omega}$$

$$+ \hat{M}_{\alpha\beta\gamma\delta\mu}\ . \tag{8.29}$$

Comparing eq. (8.28) and eq. (8.29) with eqs.(8.20,23) and eq. (8.21) we see that all the lower order terms in eq. (8.28) and eq. (8.29) are contained in the tensors $\hat{L}_{\alpha\beta\gamma\delta}$ and $\hat{M}_{\alpha\beta\gamma\delta\mu}$ respectively. Furthermore, $\hat{L}_{\alpha\beta\gamma\delta}$ and $\hat{M}_{\alpha\beta\gamma\delta\mu}$ are zero in flat space.

A pictorial representation of what we shall eventually show is

$$\begin{tikzcd} \overset{0}{\{Q_{\alpha\beta}\}} \arrow[r] & \overset{1}{\{L_{\alpha\beta\gamma}\}} \arrow[r] & \overset{2}{\{M_{\alpha\beta\gamma\delta},\ H_{\alpha\beta}\}} \arrow[r] \arrow[ll, bend left] & \overset{3}{\{H_{\alpha\beta\gamma}\}} \arrow[r] & \overset{4}{\{I_{\alpha\beta}\}} \arrow[llll, bend left] \end{tikzcd} , \tag{8.30}$$

where the numbers 0,1,2,3,4 label the order of the terms involved, and an arrow shows the result on the order of a term after one covariant differentiation. For example, covariantly differentiating $M_{\alpha\beta\gamma\delta}$ gives lower order terms involving $Q_{\alpha\beta}$, and $L_{\alpha\beta\gamma}$, and the higher order terms $H_{\alpha\beta\gamma}$.

We shall say that x equals y, modulo lower order terms, if both x and y belong to S_a, for some a, and $(x-y)$ is a linear sum of terms of order less than a, and whose coefficients are concomitants of the metric tensor, the Riemann tensor and its covariant derivatives.

Definition: A set S is spanned by $\{T,..U\}$ iff $S \subset S_a$ for some a, and, for all x belonging to S, x is, modulo lower order terms, a linear sum of T,..U, whose coefficients are concomitants of the metric tensor, the Riemann tensor, and its covariant derivatives.

Lemma K3: The set S_1 is spanned by the set $\{L_{\alpha\beta\gamma}\}$.

Eqs.(8.17,10,24) establish this lemma, while eqs.(8.20,10,25,7) prove

Lemma K4: The set S_2 is spanned by the set $\{M_{\alpha\beta\gamma\delta}, H_{\alpha\beta}\}$.

Lemma K5: The set S_3 is spanned by the set $\{H_{\alpha\beta\gamma}\}$.

Proof: From Lemma K2, it is sufficient to show that the set $\{H_{\alpha;\beta;\gamma}\}$ is spanned by the set $\{H_{\alpha\beta\gamma}\}$.

From eq.(8.29), and the covariant derivative of eq.(8.28),

$$L_{\alpha\beta\gamma;\delta;\mu} = \tfrac{1}{2}\left(\delta^{\phi\psi}_{\alpha\beta}\ \delta^{\omega\chi}_{\gamma\delta} + \delta^{\phi\psi}_{\gamma\delta}\ \delta^{\omega\chi}_{\alpha\beta}\right)\ \Theta_{\phi\chi\mu;\psi\omega}$$

$$+ 2\Theta_{\gamma\delta[\beta;\alpha];\mu} + \hat{L}_{\alpha\beta\gamma\delta;\mu} + \hat{M}_{\alpha\beta\gamma\delta\mu}\,. \qquad (8.31)$$

We shall inner product eq.(8.31) with $g^{\alpha\gamma}$, and use eqs.(8.24,10) and the Riemann identities to obtain

$$3nH_{(\beta;\delta;\mu)} + 3g_{(\beta\delta}H_{\mu);}{}^{\alpha}{}_{;\alpha} - 3H^{\alpha}{}_{;\alpha(;\mu}g_{\beta\delta)} \qquad (8.32a)$$

$$= 2\hat{L}_{\alpha\beta}{}^{\alpha}{}_{\delta;\mu} + 2\hat{M}_{\alpha\beta}{}^{\alpha}{}_{\delta\mu} + (n+1)\ R_{\mu(\beta\delta)\theta}H^{\theta}$$

$$- H_{\theta}R^{\theta}{}_{(\delta\beta)\mu}\,, \qquad (8.32b)$$

$$= 2\hat{A}_{(\beta\delta\mu)}\,, \qquad (8.32c)$$

where the term $\hat{A}_{(\beta\delta\mu)}$ is defined by eq.(8.32b). Next, inner product $g^{\beta\delta}$ into eq.(8.32b), and use $2g^{\beta\gamma}\ H_{[\alpha;\beta];\gamma}$ to obtain

$$(n-2)\ H_{\beta;}{}^{\beta}{}_{;\mu} + 2(n+1)\ H_{\mu;}{}^{\beta}{}_{;\beta} = 2\hat{L}_{\alpha\beta}{}^{\alpha\beta}{}_{;\mu}$$

$$+ 2\hat{M}_{\alpha\beta}{}^{\alpha\beta}{}_{\mu} - 2H_{\theta}R^{\theta}{}_{\mu}\,, \qquad (8.33)$$

$$H_{\alpha;\beta;}{}^{\beta} - H^{\beta}{}_{;\beta;\alpha} = 2H_{[\alpha;\beta];\gamma}\,g^{\beta\gamma} - R_{\alpha\beta}\,H^{\beta}\,. \qquad (8.34)$$

Eqs.(8.33,34) form a linear system of equations for $H_{\beta;}{}^{\beta}{}_{;\alpha}$ and $H_{\alpha;}{}^{\beta}{}_{;\beta}$, expressing either as a function of lower order terms, and $H_{[\alpha;\beta];\gamma}$. This statement, together with eq.(8.32b), the identity

$$H_{\alpha;\beta;\gamma} = H_{(\alpha;\beta;\gamma)} + \tfrac{2}{3}\,[H_{[\alpha;\beta];\gamma} + H_{[\alpha;\gamma];\beta}]$$

$$+ \tfrac{1}{3}\,(R_{\beta\gamma\alpha\delta} + R_{\alpha\beta\gamma\delta})\,H^{\delta}\,,$$

and the definition in eq.(8.8) proves Lemma K5.

When considering the fourth derivatives of $Q_{\alpha\beta}$, or the set S_4, it is sufficient to consider only the terms $H_{[\alpha;\beta];\gamma;\delta}$. If we covariant differentiate eq.(8.32) with respect to ε, and anti-symmetrize over μ and ε, we find

$$2nH_{[\mu;\varepsilon];\delta;\beta} + 3g_{(\beta\delta}H_{\mu);}{}^{\alpha}{}_{;\alpha;\varepsilon} - 3g_{(\beta\delta}H_{\varepsilon);}{}^{\alpha}{}_{;\alpha;\mu}$$

$$- 3H^{\alpha}{}_{;\alpha;(\mu;|\varepsilon|}g_{\beta\delta)} + 3H^{\alpha}{}_{;\alpha;(\varepsilon;|\mu|}g_{\beta\delta)}$$

$$= 2\hat{B}_{\mu\varepsilon\beta\delta}, \qquad (8.35)$$

where $\hat{B}_{\mu\varepsilon\beta\delta}$ involves only lower order terms. Next, covariant differentiate eq.(8.33) with respect to ε, to obtain

$$2(n+1)\, H_{\mu;}{}^{\alpha}{}_{;\alpha;\varepsilon} = (2-n)\, H^{\beta}{}_{;\beta;\mu;\varepsilon} + 2\Big[L_{\alpha\beta}{}^{\alpha\beta}{}_{;\mu;\varepsilon}$$

$$+ \hat{M}_{\alpha\beta}{}^{\alpha\beta}{}_{\mu;\varepsilon} - 2(H_{\beta}R^{\beta}{}_{\mu})_{;\varepsilon}\Big]. \qquad (8.36)$$

If we substitute eq.(8.36) into eq.(8.35), and note the definition in eq.(8.9), we have established the first part of

Lemma K6: The set S_4 is spanned by the set $\{I_{\alpha\beta}\}$. $I_{\alpha\beta}$ is trace-free, modulo lower order terms.

Proof: The second part of Lemma K6 is proved by operating on eq.(8.33) with $g^{\mu\rho}\nabla_{\rho}$, expressing $3nH^{\alpha}{}_{;\alpha;}{}^{\beta}{}_{;\beta}$ in terms of only lower order expressions.

For the fifth derivatives of $Q_{\alpha\beta}$, or the set S_5, we need to consider only terms of the form $Q^{\alpha}{}_{;\alpha;\beta;\gamma;\delta}$. By operating on both sides of eq.(8.32b) with $g^{\rho\sigma}\nabla_{\rho}\nabla_{\sigma}$, we find

$$\frac{3n}{2}H_{(\beta;\delta;\mu);}{}^{\alpha}{}_{;\alpha} = \frac{3}{2}H^{\alpha}{}_{;\alpha;(\mu;}{}^{\rho}{}_{;|\rho|}g_{\beta\delta)}$$

$$- \frac{3}{2}g_{(\beta\delta}H_{\mu);}{}^{\alpha}{}_{;\alpha;}{}^{\rho}{}_{;\rho} + \hat{A}_{(\beta\delta\mu);}{}^{\rho}{}_{;\rho}. \quad (8.37)$$

By operating on eq. (8.33) with $g^{\rho\sigma}\nabla_\rho\nabla_\sigma$, and noting the second part of Lemma K6, the right hand side of eq. (8.37) may be shown to involve only lower order terms, while the left hand side of eq. (8.37) may be written as $[3n(2-n)/4(n+1)]H^{\alpha}{}_{;\alpha;\beta;\delta;\mu}$, modulo lower order terms. This proves

Lemma K7: The set S_5 is spanned by the set $\{I_{\alpha\beta}\}$, when $n \geqslant 3$.

Theorem K1: Let R be a Riemannian space of dimension n, where $n \geqslant 3$. Then in R, the dimension of the real vector space formed from trace-free, second order conformal Killing tensors is at most N_n, where $N_n = (n-1)(n+2)(n+3)(n+4)/12$.

Proof: We shall define the row vector, Q, through the equation

$$Q = (Q_{\alpha\beta}, L_{\alpha\beta\gamma}, M_{\alpha\beta\gamma\delta}, H_{\alpha\beta}, H_{\alpha\beta\gamma}, I_{\alpha\beta}) . \tag{8.38}$$

Although the elements of Q are tensors, they can of course be displayed as some linear array. Their explicit ordering in Q is unimportant. Because of the relationships $g^{\alpha\beta}I_{\alpha\beta} = 0$, modulo lower order terms, $g^{\alpha\beta}Q_{\alpha\beta} = 0$, $L_{[\alpha\beta\gamma]} = 0$, $M_{\alpha[\beta\gamma\delta]} = 0$ and $H_{[\alpha\beta\gamma]} = 0$, the tensors $Q_{\alpha\beta}, L_{\alpha\beta\gamma}, M_{\alpha\beta\gamma\delta}, H_{\alpha\beta}, H_{\alpha\beta\gamma}$ and $I_{\alpha\beta}$ contribute $\frac{1}{2}(n-1)(n+2)$, $\frac{1}{3}(n-1)n(n+1)$, $\frac{1}{12}(n-1)n^2(n+1)$, $\frac{1}{2}(n-1)n$, $\frac{1}{3}(n-1)n(n+1)$ and $\frac{1}{2}(n-1)(n+2)$ elements respectively to Q. Their sum equals $(n-1)(n+2)(n+3)(n+4)/12$.

For $n \geqslant 3$, the preceding lemmas of this chapter establish the existence of an N_n by N_n matrix, A, which satisfies

$$\nabla_\alpha Q = QA . \tag{8.39}$$

Q is the row vector defined in eq. (8.38), $\nabla_\alpha Q$ is the covariant derivative of the elements of Q with respect to x^α, and the term QA in eq. (8.39) is the result obtained by left multiplying the matrix A by the matrix Q. The elements of A are concomitants of the metric tensor, the Riemann tensor and its covariant derivatives. Any C.K.T. is then a linear sum of the fundamental solutions of eq.(8.39), proving Theorem K1.

Having established an upper bound for the number of linearly independent C.K.T.'s in a space-time, it is natural to enquire whether this bound is attained in flat space. In Appendix E we prove

Theorem K2: Let F be a flat Riemannian space of dimension n, where $n \geqslant 3$. Then in F, the dimension of the real vector space formed from trace-free, second order conformal Killing tensors is precisely N_n, where $N_n = (n-1)(n+2)(n+3)(n+4)/12$.

CHAPTER IX

C.K.T.'S IN QUASI-DIAGONAL SPACES

In this chapter we shall consider four dimensional spaces. To make contact with our earlier work on Type D geometries, and to progress with the difficult problem of actually solving for a C.K.T., we shall also assume that our metric is quasi-diagonal, and that the C.K.T. is independent of the ignorable coordinates, x^a,

$$\mathcal{L}_{\partial_a} Q^{\alpha\beta} = 0 \,, \quad g^{a\mathfrak{a}} = 0 \,, \tag{9.1}$$

where we use the notation of eq.(5.19). It will be seen later that these conditions are satisfied by the C.K.T. found by Walker and Penrose. Since many statements about C.K.T.'s imply similar ones for Killing tensors, we shall ignore the trace-free property of C.K.T.'s.

From eq.(8.1) and eq.(9.1), the equations to be solved,

$$g^{\mathfrak{a}(\mathfrak{b}}\partial_{\mathfrak{a}} Q^{\mathfrak{c}\mathfrak{d})} - Q^{\mathfrak{a}(\mathfrak{b}}\partial_{\mathfrak{a}} g^{\mathfrak{c}\mathfrak{d})} = Q^{(\mathfrak{b}} g^{\mathfrak{c}\mathfrak{d})} \,, \tag{9.2a}$$

$$g^{\mathfrak{a}\mathfrak{b}}\partial_{\mathfrak{a}} Q^{ab} - Q^{\mathfrak{a}\mathfrak{b}}\partial_{\mathfrak{a}} g^{ab} = Q^{\mathfrak{b}} g^{ab} \,, \tag{9.2b}$$

$$- Q^{\mathfrak{a}(a}\partial_{\mathfrak{a}} g^{bc)} = Q^{(a} g^{bc)} \,, \tag{9.3a}$$

$$g^{\mathfrak{a}(\mathfrak{b}}\partial_{\mathfrak{a}} Q^{\mathfrak{c}a)} - Q^{\mathfrak{a}(\mathfrak{b}}\partial_{\mathfrak{a}} g^{\mathfrak{c}a)} = \tfrac{1}{3} Q^a g^{\mathfrak{b}\mathfrak{c}} \,, \tag{9.3b}$$

split into two sets, since eqs.(9.2) involve only $Q^{\mathfrak{a}\mathfrak{b}}$, Q^{ab}, $Q^{\mathfrak{a}}$; while eqs.(9.3) involve only $Q^{\mathfrak{a}a}$, Q^a. This simplifies the problem considerably. We shall eventually show that eq.(9.2) can be solved completely, and that no non-redundant solutions to eq.(9.3) exist unless the two vectors $Q^{3\mathfrak{a}}$ and $Q^{4\mathfrak{a}}$ are conformal Killing vectors for the two metric, $g^{\mathfrak{a}\mathfrak{b}}$.

Case 1: The Q^{ab} and $Q^{\alpha\beta}$ components.

We begin by solving eq.(9.2a), which represents the general solution for a C.K.T. in a two dimensional space. Our non-ignorable coordinates, x^a, are chosen so that the pair (g^{ab}, Q^{ab}) takes its canonical form.

The eigenvalue equation, $(Q^a{}_b - \lambda\delta^a_b)\, y^b = 0$, has as solutions

Case 1a: $\lambda, \bar{\lambda}$ complex eigenvalues; $y^a, \overline{y^a}$ eigenvectors,

Case 1b: λ_1, λ_2 real distinct eigenvalues; y_1, y_2 eigenvectors,

Case 1c: λ a double eigenvalue; y a real eigenvector.

We shall assume that Q^{ab} is non-zero, since otherwise the only solutions of eqs.(9.2) are with $Q^{\alpha\beta}$ as the symmetrized products of the two Killing vectors. In Case 1a and Case 1b the eigenvectors are linearly independent, and will be chosen as our coordinates.

Case 1a: Let us use complex coordinates, z^a, such that $\delta^a_1 = y^a$, $\delta^a_2 = \overline{y^a}$. Then Q^{ab} and g^{ab} are diagonal, and eq.(9.2a) becomes

$$g^{11}\partial_1 Q^{11} - Q^{11}\partial_1 g^{11} = Q^1 g^{11}\ , \tag{9.4a}$$

$$g^{22}\partial_2 Q^{11} - Q^{22}\partial_2 g^{11} = Q^2 g^{11}\ , \tag{9.4b}$$

$$g^{11}\partial_1 Q^{22} - Q^{11}\partial_1 g^{11} = Q^1 g^{22}\ , \tag{9.4c}$$

$$g^{22}\partial_2 Q^{22} - Q^{22}\partial_2 g^{22} = Q^2 g^{22}\ . \tag{9.4d}$$

We shall rewrite eq.(9.4a) and eq.(9.4d) as

$$Q^1 = g^{11}\partial_1(Q^{11}/g^{11})\ , \quad Q^2 = g^{22}\partial_2(Q^{22}/g^{22})\ . \tag{9.5}$$

By defining two functions, a and b, through the equations

$$Q^{11} = bg^{11}\ , \quad Q^{22} = -\ ag^{22}\ , \tag{9.6}$$

and using eq. (9.4b), we see that $\partial_2[(a+b)g^{11}] = 0$, i.e.

$$g^{11} = R_1/(a+b)\ , \quad Q^{11} = bR_1/(a+b)\ ,$$

where R_1 is an analytic function of z^1. We have assumed that $a+b \neq 0$, otherwise eq. (9.2) yields

$$Q^{\alpha\beta} = bg^{\alpha\beta} + c_1\delta_3^\alpha\delta_3^\beta + c_2\delta_3^{(\alpha}\delta_4^{\beta)} + c_3\delta_4^\alpha\delta_4^\beta\ , \tag{9.7}$$

where c_1, c_2, c_3 are complex constants, and b is a function of z^1, z^2. This is a redundant C.K.T. If $Q^{\alpha\beta}$ is a Killing tensor, then b is a complex constant, and $Q^{\alpha\beta}$ is again redundant.

Ignoring the possibility that $a+b = 0$, we find that $g^{22} = X_2/(a+b)$ and $Q^{22} = -\ aX_2/(a+b)$, where X_2 is an analytic function of z^2. Since the metric is real, $g^{11} = \overline{g^{22}}$, and so

$$g^{ab} = \frac{1}{a+\bar{a}}\begin{pmatrix} R_1 & 0 \\ 0 & \bar{R}_1 \end{pmatrix}\ , \quad X_2 = \overline{R_1}\ , \quad b = \bar{a}\ .$$

Eqs. (9.2a) are now satisfied and it remains to solve eqs. (9.2b). Those for Q^{ab} may be written as

$$\partial_1[Q^{cd} - bg^{cd}] = 0\ ,$$

$$\partial_2[Q^{cd} + ag^{cd}] = 0\ ,$$

or

$$g^{cd} = e_1{}^{cd} + f_2{}^{cd}/(a+b)\ , \ Q^{cd} = (be_1{}^{cd} - af_2{}^{cd})/(a+b)\ ,$$

where $e_1{}^{cd}$ and $f_2{}^{cd}$ are analytic functions of z^1 and z^2 respectively. Since g^{cd} is real, $\overline{f_2{}^{cd}} = e_1{}^{cd}$. The metric tensor and C.K.T. are then

$$G = g^{\alpha\beta}\partial_{(\alpha}\otimes\partial_{\beta)} = [R_1\partial_1\otimes\partial_1 + X_2\partial_2\otimes\partial_2 + (c_1+d_2)\partial_3\otimes\partial_3$$

$$+ 2\ (e_1+f_2)\partial_{(3}\otimes\partial_{4)} + (g_1+h_2)\partial_4\otimes\partial_4]/(a+b)\ , \qquad (9.8)$$

$$Q = Q^{\alpha\beta}\partial_{(\alpha}\otimes\partial_{\beta)} = [bR_1\partial_1\otimes\partial_1 - aX_2\partial_2\otimes\partial_2 + (bc_1-ad_2)\partial_3\otimes\partial_3$$

$$+ 2(be_1-af_2)\partial_{(3}\otimes\partial_{4)} + (bg_1-ah_2)\partial_4\otimes\partial_4]/(a+b)\ ,$$

$$Q^1 = g^{11}\partial_1 b\ , \quad Q^2 = -g^{22}\partial_2 a\ , \qquad (9.9)$$

where R_1, c_1, e_1, g_1 and X_2, d_2, f_2, h_2 are analytic functions of z^1 and z^2 respectively, and $\overline{b} = a$, $X_2 = \overline{R_1}$, $d_2 = \overline{c_1}$, $f_2 = \overline{e_1}$ and $h_2 = \overline{g_1}$. If Q is a Killing tensor, then a and b are functions of z^1 and z^2 respectively, and there is essentially no difference between the given Killing tensor and the metric, since the Killing tensor in eq. (9.9) may be written in the form of the metric in eq. (9.8) by an obvious redefinition of the functions $a, b, c_1, d_2, e_1, f_2, R_1, X_2$; and interchanging the Killing vectors. In fact the symmetric form of the metric and the Killing tensor in eq. (8.1), when Q^{α} is zero, shows that if $Q^{\alpha\beta}$ is regarded as the metric, then $g^{\alpha\beta}$ is its Killing tensor.

The C.K.T. in eq. (9.9) is redundant (†) when there exists constants j_1, j_2, j_3, and a function, ϕ, of z^1, z^2 such that

$$Q^{\alpha\beta} + \phi g^{\alpha\beta} + j_1\delta^{\alpha}_{3}\delta^{\beta}_{3} + j_2\delta^{(\alpha}_{3}\delta^{\beta)}_{4} + j_3\delta^{\alpha}_{4}\delta^{\beta}_{4} = 0\ . \qquad (9.10)$$

We see from the $(\mathfrak{a}\mathfrak{b})$ components that $R_1(1+\phi b) = 0 = X_2(1-\phi a)$, and as R_1 and X_2 are non-zero, we conclude that $\phi(a+b) = 0$. Consequently, whenever the C.K.T. in eq. (9.9) is defined, it is non-redundant.

The metric tensor in eq. (9.8) can be written in terms of real

(†) Assuming there are no extra conformal Killing vectors present.

coordinates by defining $z^1 = x^1 + ix^2$, where x^1 and x^2 are real. We find that

$$G = [A\partial_1 \otimes \partial_1 - A\partial_2 \otimes \partial_2 + 2B\partial_{(1} \otimes \partial_{2)} + C\partial_3 \otimes \partial_3$$

$$+ 2D\partial_{(3} \otimes \partial_{4)} + E\partial_4 \otimes \partial_4]/\psi, \qquad (9.11)$$

where A,B,C,D,E are real harmonic functions of x^1 and x^2; and ψ is harmonic iff the C.K.T. is a Killing tensor. The corresponding Q in eq.(9.9) is pure imaginary, but as any C.K.T. is only defined up to a multiplicative constant, we can always make Q real.

Case 1b: Let us choose our coordinates so that the linearly independent real eigenvectors are δ^μ_1 and δ^μ_2. Then g^{ab} and Q^{ab} are diagonal, and the preceding analysis of Case 1a applies, so that the metric and C.K.T. are given in eq.(9.8) and eq.(9.9) respectively, but now R_1, c_1, e_1, g_1 and X_2, d_2, f_2, h_2 are (not necessarily analytic) real functions of x^1 and x^2 respectively.

Case 1c: By diagonalizing g^{ab} and writing out the condition for a double eigenvalue, $(g^{11}Q_{11} - g^{22}Q_{22})^2 + 4g^{11}g^{22}Q_{12}^2 = 0$, we see that either $Q^{12} = 0$, in which case $Q^{\alpha\beta}$ is redundant, or signature $(g^{ab}) = 0$. Assuming the latter, we may choose as our non-ignorable coordinates the two real null directions. Then $g^{ab} = 2g^{12}\delta^{(a}_1\delta^{b)}_2$, and the condition for a double eigenvalue is $Q^{11}Q^{22} = 0$. Without loss of generality we shall set $Q^{11} = 0$. If $Q^{22} = 0$ also, then Q^{ab} is redundant, and we have the solution of eq.(9.7), so we shall assume that $Q^{22} \neq 0$. Our canonical forms are then

$$g^{ab} = \begin{pmatrix} 0 & g^{12} \\ g^{12} & 0 \end{pmatrix}, \quad Q^{ab} = \begin{pmatrix} 0 & Q^{12} \\ Q^{12} & Q^{22} \end{pmatrix}, \quad Q^{22} \neq 0, \qquad (9.12)$$

and the coordinate freedom is $x^{2'} = x^{2'}(x^2)$, $x^{1'} = x^{1'}(x^1)$.

Eq. (9.2a) may be written as

$$g^{12}\partial_2 Q^{12} - Q^{12}\partial_2 g^{12} = Q^1 g^{12} , \tag{9.13a}$$

$$g^{12}\partial_2 Q^{22} + 2g^{12}\partial_1 Q^{12} - 2Q^{22}\partial_2 g^{12} - 2Q^{12}\partial_1 g^{12} = 2Q^2 g^{12} , \tag{9.13b}$$

$$\partial_1 Q^{22} = 0 , \tag{9.13c}$$

and so the coordinate transformation $x^{2'} = x^{2'}(x^2)$ will be used to set $Q^{22} = \pm 1$. From eq. (9.13a) and eq. (9.13b),

$$Q^1 = g^{12}\partial_2 (Q^{12}/g^{12}) ,$$

$$Q^2 + x^2 g^{12}\partial_1 (Q^1/g^{12}) = g^{12}\partial_2 [x^2\partial_1 (Q^{12}/g^{12}) + Q^{22}/g^{12}] , \tag{9.14}$$

which suggests defining functions a and b as

$$Q^{12} = ag^{12} ,$$

$$b = x^2\partial_1 a + (Q^{22}/g^{12}) , \tag{9.15}$$

and so, from eq. (9.14), a and b are functions of x^1 iff Q is a Killing tensor. Since Q^{22} is non-zero, so is $b - x^2\partial_1 a$. The remaining equations are

$$g^{12}\partial_2 Q^{cd} - Q^{12}\partial_2 g^{cd} = Q^1 g^{cd} , \tag{9.16a}$$

$$g^{12}\partial_1 Q^{cd} - Q^{12}\partial_1 g^{cd} - Q^{22}\partial_2 g^{cd} = Q^2 g^{cd} . \tag{9.16b}$$

Substituting $Q^{cd} = ag^{cd} + e_1{}^{cd}$ into eq. (9.16a) shows that $e_1{}^{cd}$ is function of x^1 alone, while eq. (9.16b) implies that

$$g^{cd} = (x^2\partial_1 e_1{}^{cd} + f_1{}^{cd})/(b - x^2\partial_1 a) ,$$

$$Q^{cd} = ag^{cd} + e_1{}^{cd} ,$$

where f_1^{cd} is a function of x^1 alone. The complete solution for the metric and C.K.T. is then

$$G = [2Q^{22}\partial_{(1}\otimes\partial_{2)} + (x^2\partial_1 e_1^{cd} + f_1^{cd})\,\partial_{(c}\otimes\partial_{d)}] \,/\, (b - x^2\partial_1 a)\,, \tag{9.17}$$

$$Q = aG + Q^{22}\partial_2\otimes\partial_2 + e_1^{cd}\partial_{(c}\otimes\partial_{d)}\,, \tag{9.18}$$

where e_1^{cd}, f_1^{cd} are functions of x^1, $Q^{22} = \pm 1$, and Q is a Killing tensor iff a and b are functions of x^1. This completes the solution of eqs.(9.2) and it remains to discuss eq.(9.3).

<u>Case 2</u>: The $Q^{a\mathfrak{a}}$ components.

If the vectors ξ^α and η^α are defined through the equations $\xi^\alpha\partial_\alpha = 2Q^{3\mathfrak{a}}\partial_{\mathfrak{a}}$ and $\eta^\alpha\partial_\alpha = 2Q^{4\mathfrak{a}}\partial_{\mathfrak{a}}$, then $Q^{\alpha\beta} = \xi^{(\alpha}\delta_3^{\beta)} + \eta^{(\alpha}\delta_4^{\beta)}$, and since δ_a^α are Killing vectors, the condition for $Q^{a\mathfrak{a}}$ to be a C.K.T. is

$$\xi^{(\alpha;\beta}\delta_3^{\gamma)} + \eta^{(\alpha;\beta}\delta_4^{\gamma)} = Q^{(\alpha}g^{\beta\gamma)}\,.$$

Consequently,

$$Q^{\mathfrak{a}} = 0\,,\quad \xi^{(\mathfrak{a};\mathfrak{b})} = Q^3 g^{\mathfrak{a}\mathfrak{b}}\,,\quad \eta^{(\mathfrak{a};\mathfrak{b})} = Q^4 g^{\mathfrak{a}\mathfrak{b}}\,,$$

$$2\xi^{(3;4)} + \eta^{(3;3)} = 2Q^3 g^{34} + Q^4 g^{33}\,,$$

$$\xi^{(4;4)} + 2\eta^{(3;4)} = Q^3 g^{44} + 2Q^4 g^{34}\,,$$

$$\xi^{(3;3)} = Q^3 g^{33}\,,\quad \eta^{(4;4)} = Q^4 g^{44}\,,$$

and so $Q^{a\mathfrak{a}}$ is formed from symmetrized products of the Killing vectors δ_a^α and the two dimensional conformal Killing vectors ξ^α and η^α. If either η^α or ξ^α is zero, then the other is a conformal Killing vector for the whole space. If $\eta^\alpha = f\xi^\alpha$, (i.e. $\det.Q^{a\mathfrak{a}} = 0$), then either f is a constant, in which case a linear transformation among the ignorable coordinates, x^a, will set $Q^{3\mathfrak{a}}$ to zero, or f is non-constant. This is only possible when ξ^α and η^α are null. Substituting this information into eq. (9.3a) implies that either $g^{ab} = 0$, or f is a constant. Consequently, we assume that $\det.(Q^{a\mathfrak{a}}) \neq 0$.

Specialized solutions of eq. (9.3) may be found; for example

$$G = \phi\{G_0{}^{\mathfrak{a}\mathfrak{b}}\partial_{(\mathfrak{a}} \otimes \partial_{\mathfrak{b})} + (a(x^2)^2 + 2cx^2 + d)\partial_3 \otimes \partial_3$$

$$- 2(ax^1x^2 + bx^2 + cx^1 + f)\partial_{(3} \otimes \partial_{4)} + (a(x^1)^2 + 2bx^1 + e)\partial_4 \otimes \partial_4\} ,$$

$$Q = 2\partial_{(1} \otimes \partial_{3)} + 2\partial_{(2} \otimes \partial_{4)} ; \quad Q^3 = -\partial_1 \log\phi , \quad Q^4 = -\partial_2 \log\phi . \tag{9.19}$$

where $G_0{}^{\mathfrak{a}\mathfrak{b}}$ is a constant matrix; a,b,c,d,e,f are constants; and ϕ is an arbitrary function of x^1 and x^2. However, the author was unable to solve eq. (9.3) completely, and so we shall not discuss the $Q^{a\mathfrak{a}}$ components further.

This chapter will be completed by using the C.K.T. in eq. (9.9) to solve the null geodesic equations for the metric in eq. (9.8). If our C.K.T. is a Killing tensor, the discussion applies to all geodesics.

For the metric of eq. (9.8) there are four independent first integrals; namely

$$L_1 = g^{\alpha\beta}P_\alpha P_\beta ,$$

$$L_2 = Q^{\alpha\beta}P_\alpha P_\beta ,$$

$$L_3 = \delta^\alpha_3 P_\alpha ,$$

$$L_4 = \delta^\alpha_4 P_\alpha , \tag{9.20}$$

where

$$P^\alpha = \frac{dx^\alpha}{d\lambda} , \tag{9.21}$$

is a null geodesic with an affine parameter λ, and L_1, L_2, L_3, L_4 are constant along P^{α}. From eqs.(9.8,9,20),

$$R_1(P_1)^2 = aL_1 - c_1L_3^2 - 2e_1L_3L_4 - g_1L_4^2 + L_2 ,$$

$$X_2(P_2)^2 = bL_1 - d_2L_3^2 - 2f_2L_3L_4 - h_2L_4^2 - L_2 . \qquad (9.22)$$

In the two cases when Q is a C.K.T., or a Killing tensor, we see from eq.(9.22) that P_1 and P_2 are functions of x^1 and x^2 respectively,

$$P_1 = P_1(x^1) , \quad P_2 = P_2(x^2) .$$

From eq.(9.21),

$$\frac{dx^1}{d\lambda} = R_1P_1/(a+b) ,$$

$$\frac{dx^2}{d\lambda} = X_2P_2/(a+b) ,$$

$$\frac{dx^3}{d\lambda} = [(c_1+d_2)L_3 + (e_1+f_2)L_4]/(a+b) ,$$

$$\frac{dx^4}{d\lambda} = [(g_1+h_2)L_4 + (e_1+f_2)L_3]/(a+b) , \qquad (9.23)$$

the geodesic equations are separable[13]. They reduce to ordinary differential equations, which may be solved by quadrature[11].

$$dx^1/R_1P_1 = dx^2/X_2P_2 , \qquad (9.24a)$$

$$dx^3 = [L_3c_1 + L_4e_1]dx^1/R_1P_1 + [L_3d_2 + L_4f_2]dx^2/X_2P_2 , \qquad (9.24b)$$

$$dx^4 = [L_3e_1 + L_4g_1]dx^1/R_1P_1 + [L_3f_2 + L_4h_2]dx^2/X_2P_2 , \qquad (9.24c)$$

$$d\lambda = a\,dx^1/R_1P_1 + b\,dx^2/X_2P_2 . \qquad (9.24d)$$

This solves the (null) geodesic equations when Q is a (conformal) Killing tensor. For example, $x^4 = x^4(1) + x^4(2)$, where $x^4(1) = \int(L_3e_1 + L_4g_1)dx^1/R_1P_1$; $x^4(2) = \int(L_3f_2 + L_4h_2)dx^2/X_2P_2$.

To solve eq.(9.24d) when Q is not a Killing tensor requires that we solve eq.(9.24a), and then substitute for either x^1 or x^2 as a function of x^2 or x^1 respectively. It is possible that eqs.(9.24) may be simplified by using the remaining coordinate freedom; $x^{1'} = x^{1'}(x^1)$, $x^{2'} = x^{2'}(x^2)$, $x^{a'} = c^{a'}{}_a x^a$, where $c^{a'}{}_a$ are real constants, and $x^{1'}$ and $x^{2'}$ are functions of x^1 and x^2 respectively.

Comparing eq.(5.62) and eq.(6.21) with eq.(9.8), we see that the Kinnersley and Kerr-NUT metrics are characterized by

$$\begin{aligned}
a &= 2X^2/(1-XZ)^2 , & b &= 2Z^2/(1-XZ)^2 , \\
c_1 &= -X^4/\tilde{g}(X) , & d_2 &= Z^4/\tilde{h}(Z) , \\
e_1 &= -X^2/\tilde{g}(X) , & f_2 &= -Z^2/\tilde{h}(Z) , \\
g_1 &= -1/\tilde{g}(X) , & h_2 &= 1/\tilde{h}(Z) , \\
c_1 g_1 &= e_1^2 , & d_2 h_2 &= f_2^2 ,
\end{aligned}$$

and

$$\begin{aligned}
a &= x^2 , & b &= y^2 , \\
c_1 &= -x^4/V , & d_2 &= y^4/W , \\
e_1 &= -x^2/V , & f_2 &= -y^2/W , \\
g_1 &= -1/V , & h_2 &= 1/W , \\
c_1 g_1 &= e_1^2 , & d_2 h_2 &= f_2^2 ,
\end{aligned}$$

respectively. Let us assume that the perfect square condition, $c_1 g_1 = e_1^2$ and $d_2 h_2 = f_2^2$, holds for the metric in eq.(9.8), and that $R_1, X_2, c_1, e_1, g_1, -d_2, -f_2, -h_2$ are positive on some coordinate patch, with h and g being non-zero. Redefining d,f and h by $(d,f,h) \to -(d,f,h)$ allows eq.(9.8) to be written as

$$ds^2 = (a+b)[(dx^1/\sqrt{R_1})^2 + (dx^2/\sqrt{X_2})^2 + (m_1/p_1+q_1)^2(dx^3+q_2dx^4)^2 - (n_2/p_1+q_2)^2(dx^3-p_1dx^4)^2] ,$$

$$m_1 = g_1^{-\frac{1}{2}}, \quad n_2 = h_2^{-\frac{1}{2}}, \quad p_1 = \sqrt{c_1/g_1}, \quad q_2 = -\sqrt{d_2/h_2}, \qquad (9.25)$$

where m_1, n_2, p_1, q_2 are functions of x^1, x^2, x^1, x^2 respectively.

A straight-forward calculation shows that the two real null vectors, $\omega^{0\dot{0}}$ and $\omega^{1\dot{1}}$, defined by

$$\begin{pmatrix} \omega^{0\dot{0}} \\ \omega^{1\dot{1}} \end{pmatrix} = dx^2/\sqrt{X_2} \pm (n_1/p_1 + q_2)(dx^3 - p_1 dx^4) ,$$

are geodesic and shearfree. Consequently, the empty space subclass of eq. (9.25) with a diverging p.n.v. are the Type D metrics. The corresponding C.K.T. is that found by Walker and Penrose.

SUMMARY AND CONCLUSION

The empty space Type D metrics with a diverging ray congruence have been derived in a coordinate system used by [K.D.]. The radiating metrics have a complex Killing vector, ∂_ζ, while the non-radiating metrics have a real Killing vector, $P\partial_u$ [32]. These latter metrics are a well-defined limit of the radiating Kinnersley metric.

It was shown that the diverging, empty, Type D spaces are precisely those diverging, empty, algebraically special spaces which are quasi-diagonal. This fact was used to derive the canonical form, discovered by J. Plebanski and M. Demianski, for the Type D metrics.

The empty Type D spaces are a natural subclass of the algebraically special spaces with two commuting Killing vectors. The latter spaces are characterized by four real constants, n_1, n_2, μ_1, μ_2. The space is Type D iff n_1 and n_2 are zero. This reveals an asymmetry in the results known at present for the algebraically special, empty spaces with two commuting Killing vectors. In particular, all non-radiating spaces with $n_1 = 0$ are known, but only those with both n_1 and n_2 zero are known for the radiating subclass. Whether this larger class of radiating metrics can be found is an interesting unanswered question.

To find the canonical form, eq. (5.62), for the Kinnersley metric, it was necessary to transform from the affine parameter along the principal null congruence to a new coordinate. Most of the work on algebraically special spaces incorporates this affine parameter into its mathematical formalism in an essential way.

This dissertation shows that this choice of coordinate for the radiating Type D metrics does not lead to simple metrical forms, suggesting that further thought be given to the choice of coordinates in an algebraically special, empty space.

Possibly the most surprising result of this thesis is that the radiating metrics are, in a "structural sense", conformally related to the non-radiating metrics. This fact was the starting point for J. Plebanski and M. Demianski, and the conclusion eventually reached by R.P. Kerr and myself. At present there are no known radiating and twisting, empty Type III metrics[33]; and only one such Type N metric is known[34]. Whether solutions in these Petrov classes can be found by looking at the "conformal equivalent" to some non-radiating, twisting metric remains to be seen. However, such an approach relies heavily on the initial choice of coordinates, and so the problem reduces to finding a preferred set of coordinates in the space.

In Chapter 9 we showed that the q.d. metrics with non-redundant q.d. C.K.T.'s split into three classes. Two of these have a two metric, g^{ab}, with zero signature, while the other has an arbitrary signature. This latter class is characterized by two quadratic functions, and when these reduce to two perfect squares, we have the metrical forms of Carter[23]. His work was applied to the class containing the empty Type D metrics, and it was shown that the null geodesics may be found by quadrature. Much work remains to be done on C.K.T.'s.

APPENDIX [K.D.]

This appendix summarizes part of [K.D.]'s paper. Let Φ be a geometric object. Then if $x \to x^*(x,t)$ is a one-parameter set of coordinate transformations, the Lie derivative of Φ, with respect to a vector K, is

$$\mathcal{L}_K \Phi = \frac{d}{dt}\left(\Phi^*(x^*) - \Phi(x^*)\right)\Big|_{t=0}, \tag{K.D.1}$$

where

$$K = \partial_t\big|_{t=0} = \left(\frac{\partial x^{\mu^*}}{\partial t}\Big|_{t=0}\right)\partial_\mu . \tag{K.D.2}$$

From eq.(3.10), eq.(3.17) and eq.(3.21a), the one-parameter coordinate transformations to be considered are

$$\begin{aligned} \zeta^* &= \psi(\zeta,t) , \\ u^* &= |\psi_\zeta|(u + \mathcal{U}(\zeta,\bar{\zeta},t)) , \\ v^* &= |\psi_\zeta|^{-1} v , \\ \psi(\zeta,0) &= \zeta ,\quad \mathcal{U}(\zeta,\bar{\zeta},0) = 0 . \end{aligned} \tag{K.D.3}$$

From eq.(K.D.2) and eq.(K.D.3), we have

$$K = \alpha\partial_\zeta + \bar{\alpha}\partial_{\bar{\zeta}} + \mathrm{Re}(\alpha_\zeta)(u\partial_u - v\partial_v) + P\partial_u , \tag{K.D.4}$$

where

$$\alpha = \psi_t\big|_{t=0}, \quad P = \mathcal{U}_t\big|_{t=0} .$$

The discussion in the second paragraph of Chapter IV showed that Killings' equations, $\mathcal{L}_K g^{\alpha\beta} = 0$, reduce to $\mathcal{L}_K \Omega = 0 = \mathcal{L}_K M$. These equations can be calculated using eq.(K.D.1), and the transformation properties of Ω and M, given in eq.(3.18) and eq.(3.25) respectively. For example,

$$0 = -\not{\mathcal{L}}_K\Omega = -\frac{d}{dt}\{(|\psi_\zeta|/\psi_\zeta)[\Omega - \mathcal{U}_\zeta - (\log|\psi_\zeta|)_\zeta(u+\mathcal{U}) - \Omega(x^*)]\}$$

$$= K\Omega + P_\zeta + \tfrac{1}{2}u\alpha_{\zeta\zeta} + \tfrac{1}{2}(\alpha_\zeta - \overline{\alpha}_{\overline{\zeta}})\,\Omega\,.$$

APPENDIX A

We shall show that choosing $\lambda_0 = -i$ in eq.(5.8) yields a simpler result than $\lambda_0 = 1$. Eq.(5.8) becomes

$$\dot{\theta}^2 - i\ddot{\theta} = -i\,e^{2i(\theta + \frac{\pi}{4})}/2 . \tag{A.1}$$

Defining a new coordinate X, and a new variable T, by

$$\frac{dX}{dx} = (1 + X^4)^{\frac{1}{2}}/2 , \quad T = \tan(\theta + \tfrac{\pi}{4}) , \tag{A.2}$$

allows us to write the real part of eq.(A.1) as

$$\frac{d\sqrt{T}}{\sqrt{1 + (\sqrt{T})^4}} = \frac{dX}{\sqrt{1 + X^4}} = \frac{dx}{2} . \tag{A.3}$$

We still have the freedom $x \to x + x_0$, and so we can take any particular solution of eq.(A.3) as the general solution, by fixing the real constant x_0. We shall choose

$$T = X^2 , \tag{A.4}$$

and define

$$\tilde{g} = g(1 + X^4) \; ; \quad \partial_X \tilde{g} = \dot{\tilde{g}} . \tag{A.5}$$

Eq.(5.6) may then be written as

$$\tfrac{1}{4}(X^2 + X^5)\,\ddot{\tilde{g}} - \tfrac{1}{4}(5X^4 + 1)\,\dot{\tilde{g}} + 2X^3\,\tilde{g}$$
$$= m(X^6 - 3X^2) + n(3X^4 - 1) , \tag{A.6}$$

which has the general solution

$$\tilde{g} = a(X^4 - 1) + bX^2 - 4mX^3 + 4nX , \tag{A.7}$$

where

$$m + in = \mu_0 e^{-3i\pi/4} . \tag{A.8}$$

The function, Ω, is given by

$$\Omega = g^{-\frac{1}{2}} e^{i\theta} = 2X^2 / (\tilde{g}) + i(X^4 - 1) / \tilde{g} . \tag{A.9}$$

APPENDIX B

We shall investigate when an $\mathfrak{A}(2,R)$ is q.d. From eq.(5.22a), two functions, f^A, will quasi-diagonalize the metric iff

$$f^A{}_{,[\mathfrak{a},\mathfrak{b}]} = \left(h^{AB}\, g_{B[\mathfrak{a},}\right)_{\mathfrak{b}]} = 0\,. \tag{B.1}$$

The computations in eq.(B.1) are simplified by noting the following results of Papapetrou. He has shown that if we define

$$A_\lambda = \epsilon_{\lambda\mu\nu\rho}\, a^\mu \,(\sqrt{-g}\; a^\nu)^{;\rho}\,, \tag{B.2}$$

$$B_\lambda = \epsilon_{\lambda\mu\nu\rho}\, b^\mu \,(\sqrt{-g}\; b^\nu)^{;\rho}\,, \tag{B.3}$$

$$g = \det(g_{\alpha\beta})\,, \tag{B.4}$$

where $a^\mu = \delta^\mu_3$ and $b^\mu = \delta^\mu_4$ are two commuting Killing vectors, and $g_{\alpha\beta}$ is an empty space metric, then

$$f^A{}_{,[1,2]} = \sqrt{-g}\;\delta^{-1}\, h^{AB}\, C_B \quad, \tag{B.5}$$

where

$$(C_3, C_4) = ((\sqrt{-g}\; a^1)^{;2}\,, \quad (\sqrt{-g}\; b^1)^{;2})\,, \tag{B.6}$$

$$\delta = g_{33}\, g_{44} - (g_{34})^2\,, \tag{B.7}$$

$$h^{AB}\, g_{BC} = \delta^A_C\,. \tag{B.8}$$

Furthermore, he has shown that, since $g_{\alpha\beta}$ is an empty space metric, the C_3 and C_4 in eq.(B.6) are constants, establishing

<u>Papapetrou's Theorem:</u> An empty space, $\mathfrak{S}$, with two commuting Killing vectors, $a^\mu = \delta^\mu_3$ and $b^\mu = \delta^\mu_4$, is quasi-diagonalizable iff the two constants $\sqrt{-g}\; a^{1;2}$ and $\sqrt{-g}\; b^{1;2}$ are both zero.

We shall prove Theorem IIIR by showing that in an $\mathcal{OL}(2,R)$ the constants C_3 and C_4 in eq.(B.6) are proportional to ν_0. Our aim, then, is to eventually evaluate $\sqrt{-g}\, K^{1;2}$.

When the Killing vector, K, is complex,

$$K = K^{\mu}\partial_{\mu} = \delta^{\mu}_{3}\partial_{\mu} = \partial_{\zeta}, \tag{B.9}$$

the necessary and sufficient condition for an $\mathcal{OL}(2,R)$ to be q.d. is that $K^{[1;2]} = 0$.

Our calculations are simplified by using a null tetrad system. This consists of four null vectors and four one-forms, defined by $e_a = e_a{}^{\mu}\partial_{\mu}$ and $\omega^a = e^a{}_{\mu}dx^{\mu}$ respectively. The latin indices belong to the set $\{1,2,3,4\}$, and are associated with the spinor indices, $A\dot{A}$, through the relations

$$1 \leftrightarrow 0\dot{0}\,, \quad 2 \longleftrightarrow 1\dot{1}\,, \quad 3 \longleftrightarrow 0\dot{1}\,, \quad 4 \longleftrightarrow 1\dot{0}\,. \tag{B.10}$$

For example, $\omega^3 = e^3{}_{\mu}dx^{\mu} = \omega^{0\dot{1}}{}_{\mu}dx^{\mu}$. The quantities $\{e_a{}^{\mu}\}$ and $\{e^a{}_{\mu}\}$ are therefore defined in eq.(3.27) and eq.(3.28) respectively.

In an $\mathcal{OL}(2,R)$, the Lie Derivative of $e_a{}^{\mu}$ with respect to a Killing vector, K, $\mathcal{L}_K e_a{}^{\mu}$, is zero.

$$\mathcal{L}_K e_a{}^{\mu} = e_a{}^{\mu}{}_{;\nu}K^{\nu} - K^{\mu}{}_{;\alpha}e_a{}^{\alpha} = 0\,. \tag{B.11}$$

The Ricci rotation coefficients[35], $\Gamma^a{}_{bc}$, are defined by

$$\Gamma^a{}_{bc} = -e^a{}_{\mu;\nu}\, e_b{}^{\mu}\, e_c{}^{\nu}, \tag{B.12}$$

and so, from eq.(B.11),

$$K^{b;a} = \Gamma^{ba}{}_{c}K^{c}, \tag{B.13}$$

where tetrad indices are raised and lowered with the tetrad metric, g^{ab}, g_{ab}.

$$g^{ab} = g_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} . \tag{B.14}$$

From eq. (B.6) and eq. (B.11),

$$K^{1;2} = (\rho\bar{\rho}) \text{ (constant)}, \tag{B.15}$$

$$= e_a{}^1 e_b{}^2 \Gamma^{abc} K_c , \tag{B.16}$$

and

$$\Gamma_{abc} = - (\epsilon_{AB} \Gamma_{\dot{A}\dot{B}C\dot{C}} + \epsilon_{\dot{A}\dot{B}} \Gamma_{ABC\dot{C}}) , \tag{B.17}$$

where $\Gamma_{ABC\dot{C}}$ are the spin coefficients, ρ is the complex divergence, and $e_a{}^1$ and $e_b{}^2$ are given in eq. (B.10) and eq. (3.28). Defining

$$T_{ab} = \Gamma_{abc} K^c , \tag{B.18}$$

allows eq. (B.16) to be written as

$$K^{1;2} = T_{12} - \rho(\iota - B\Omega)T_{13} - \bar{\rho}(\bar{\iota} + B\bar{\Omega})T_{14}$$

$$+ \rho\Omega T_{23} + \overline{\rho\Omega}\, T_{24} + \rho\bar{\rho}(\bar{\iota}\Omega - \iota\bar{\Omega})T_{34} . \tag{B.19}$$

Since $K = \partial_\zeta$, we have

$$K = -\rho^{-1}\partial_{1\dot{0}} + \Omega\partial_{1\dot{1}} + (\iota + B\Omega)\partial_{0\dot{0}} , \tag{B.20}$$

which, with eq. (B.18) and eq. (B.17), gives

$$T_{12} = -(\gamma + \bar{\gamma})\Omega + \rho^{-1}\alpha , \tag{B.21a}$$

$$T_{13} = 1 , \tag{B.21b}$$

$$T_{14} = 0 , \tag{B.21c}$$

$$T_{23} = \bar{\nu}\Omega - \rho^{-1}\bar{\mu}, \tag{B.21d}$$

$$T_{24} = \Omega\nu, \tag{B.21e}$$

$$T_{34} = (\gamma + \bar{\gamma})\Omega - \rho^{-1}\alpha, \tag{B.21f}$$

where the spin coefficients $\rho, \alpha, \gamma, \mu, \nu$ are defined in eq. (3.32). These are given below for completeness.

$$\alpha = \rho\dot{\Omega}, \tag{B.22a}$$

$$\gamma = -2M\rho^2, \tag{B.22b}$$

$$\nu = \ddot{\Omega} + \rho[\Omega\partial_u(\bar{\Omega}\ddot{\Omega}) - 2\dot{\Omega}\bar{\Omega}\ddot{\Omega} + 2i\Sigma\ddot{\Omega}] + \rho^2[-2\Omega\partial_u M - 6M\dot{\Omega}]$$

$$+ 2M\rho^3[\Omega\partial_u[\Omega\dot{\bar{\Omega}}] - \bar{\Omega}\partial_u[\Omega\dot{\Omega}]], \tag{B.22c}$$

$$\mu^{(\dagger)} = \bar{\rho}\bar{\Omega}\ddot{\Omega} - 2M\rho^2 - 2M\rho\bar{\rho}, \tag{B.22d}$$

$$B = -2M\rho - 2\overline{M\rho} + \tfrac{1}{2}\Omega\ddot{\bar{\Omega}} + \tfrac{1}{2}\bar{\Omega}\ddot{\Omega}, \tag{B.22e}$$

$$\bar{\rho} = \rho - 2i\Sigma\rho\bar{\rho}, \tag{B.22f}$$

$$\iota = \dot{\Omega}\rho^{-1} + \dot{\Omega}(\bar{\Omega}\dot{\Omega} - \Omega\dot{\bar{\Omega}}) + \tfrac{1}{2}(\Omega\bar{\Omega}\ddot{\Omega} - \Omega^2\ddot{\bar{\Omega}}). \tag{B.22g}$$

A dot, $(\dot{})$, denotes differentiation with respect to u.

Eq. (B.22), the field equations,

$$2\dot{M} = (-\bar{\Omega}\partial_u - 2\dot{\bar{\Omega}})\bar{D}\partial_u D\Omega, \tag{B.23a}$$

$$M = -\tfrac{1}{2}\mu_0\bar{\Omega}^{-3}, \tag{B.23b}$$

$$\bar{D}\partial_u D\Omega = 3\mu_0\bar{\Omega}^{-4} - \nu_0\bar{\Omega}^{-2}, \tag{B.23c}$$

(†) μ is the spin coefficient $\Gamma_{110\dot{1}}$.

$$2M - 2\overline{M} = \overline{\nu}_0 \Omega^{-1} - \nu_0 \overline{\Omega}^{-1} + \dot{\Omega}\,\dot{\overline{\Omega}}(\overline{\Omega}\,\dot{\Omega} - \Omega\,\dot{\overline{\Omega}})$$

$$+ \Omega\,\overline{\Omega}(\dot{\overline{\Omega}}\,\ddot{\Omega} - \dot{\Omega}\,\ddot{\overline{\Omega}}) \,, \tag{B.23d}$$

and eq.(B.21) may be used to find, after a lengthy calculation, that

$$K^{1;2} = -2\,\rho\overline{\rho\nu}_0 \,. \tag{B.24}$$

This calculation is simplified by noting eq.(B.15).

APPENDIX C

We shall give a brief explanation of how the coordinate transformation in eq. (5.55) may be deduced.

Papapetrou has shown that in an empty space-time the spin-vector, A_λ, associated with a Killing vector, a^α, is the gradient of a function, A.

$$A_\alpha = \epsilon_{\alpha\beta\gamma\delta}\, a^\beta \sqrt{-g}\, a^{\gamma;\delta} = A,_\alpha . \tag{C.1}$$

Since an empty Type D metric has two Killing vectors, we may use eq. (C.1) to calculate two functions for these spaces. We shall deduce eq. (5.55) by comparing these functions for the Kinnersley metric with those of the Kerr-NUT space.

Let $A,_\alpha$ and $B,_\alpha$ be the spin vectors associated with the Killing vectors $\delta^\alpha_{\eta_1}$ and $\delta^\alpha_{\eta_2}$ respectively, in the Kerr-NUT metric of eq. (6.21). Then

$$A = (m+\bar{m})x/(x^2+y^2) + i(m-\bar{m})y/(x^2+y^2)^2 + \text{constant}, \tag{C.2}$$

$$B = \cdot\, 2K_2xy - [i(m-\bar{m})x^2y^3 + (m+\bar{m})x^3y^2]/(x^2+y^2) + \text{constant} . \tag{C.3}$$

Although the functions A and B are not particularly simple, in flat space A is zero, and B is, to within a multiplicative constant, the product of the two canonical coordinates. We shall therefore look for a similar product in the functions associated with the Killing vectors in the flat space limit of the Kinnersley metric.

In the metric of eq. (5.55), we shall take $g(x) = 2\mathrm{Re}(a_0 e^{2i\theta})$. The spin vector, A_α, associated with ∂_ζ, satisfies

$$A,_x = \sqrt{2}\ ia_0\, \Sigma \sec(\theta - \phi)\, \sqrt{\cos 2\phi}\ , \tag{C.4a}$$

$$A,_y = \sqrt{2}\ ia_0\, \Sigma \sec(\theta - \phi)\, \sqrt{\cos 2\theta}\ . \tag{C.4b}$$

From eq.(5.10) and eq.(5.36c),

$$\sqrt{2}\ \partial_x = \sqrt{\cos 2\theta}\, \partial_\theta\ , \quad \sqrt{2}\ \partial_y = \sqrt{\cos 2\phi}\, \partial_\phi\ , \tag{C.5}$$

and so eq.(C.4) gives rise to the linear equation

$$\cos 2\theta\ A,_\theta - \cos 2\phi\ A,_\phi = 0\ . \tag{C.6}$$

Its characteristics are defined by

$$dA = \frac{d\theta}{\cos 2\theta} = -\frac{d\phi}{\cos 2\phi} = \text{constant}\ , \tag{C.7}$$

from which we deduce that

$$\text{constant} = \log_e \{\sqrt{\tan(\theta + \tfrac{\pi}{4})\, \tan(\phi + \tfrac{\pi}{4})}\}\ . \tag{C.8}$$

This product suggests that we apply the transformation in eq.(5.55) to the Kinnersley metric.

APPENDIX D

We shall consider those algebraically special spaces with two commuting Killing vectors, ∂_ζ and $\partial_{\bar\zeta}$. The Type D subclass has the canonical forms of eq. (5.30) and eq. (5.62), which follow from the existence of a function, y, satisfying

$$\tfrac{1}{2}\,dy = (dR + p\,dy) / \sqrt{\Sigma^2 + A^2} . \tag{D.5}$$

We shall find the integrability conditions of eq. (D.5), and then discuss what empty algebraically special spaces follow from the existence of y.

Since $p = \tfrac{1}{2}(R^2 + 3\dot\theta^2)$, $\Sigma = R^2 + \dot\theta^2$ and $A = 2R\dot\theta - 2\ddot\theta$,

we have

$$(\Sigma^2 + A^2)^{\frac{3}{2}}\, d(dy) = \{-8R\dot\theta[\dddot\theta + 2\dot\theta^3] + 8\ddot\theta[\dddot\theta + 2\dot\theta^3]\}\, dR \wedge dx . \tag{D.6}$$

From eq. (5.7), we see that the existence of y implies that the space is type D, or has $\dot\theta = 0$. This suggests we take $\theta = 0$. Defining

$$y = x - 2/R , \tag{D.7a}$$

$$d\eta_1 + i\,d\eta_2 = 2d\zeta^* - dy / g(y) , \tag{D.7b}$$

allows eq. (5.16) to be written as

$$ds^2 = \frac{-2}{(x-y)^2}\left[\frac{dx^2}{g(x)} - \frac{dy^2}{g(y)} + g(x)\,d\eta_2^2 + g(y)\,d\eta_1^2\right]$$
$$+ 2\left[\sigma - \frac{g(x) - g(y)}{(x-y)^2}\right]\left[d\eta_1 + \frac{dy}{g(y)}\right]^2 , \tag{D.8}$$

where

$$\sigma = -\mu_0 / R + \tfrac{1}{2}R\dot g - \tfrac{1}{2}\ddot g , \tag{D.9a}$$

$$\dddot{g} = -3\mu_0 + \nu_0 g^{-1} . \qquad \text{(D.9b)}$$

The p.n.v. is given by

$$K = du + \Omega d\zeta + \bar{\Omega} d\bar{\zeta} = g^{-\frac{1}{2}}(x)[\, d\eta_1 + dy \,/\, g(y)] . \qquad \text{(D.10)}$$

The metric of eq.(D.8) resembles a class of metrics discussed by Robinson and Robinson[36]. The only solutions known for eq.(D.9b) are the C metric, ($\nu_0 = 0$), and a Type III metric with $\mu_0 = 0$, and $g = 2\sqrt{-2\nu_0/3}\; x^{\frac{3}{2}}$.

APPENDIX E

Our aim is to prove Theorem K.2 by constructing the required number of linearly independent C.K.T.'s. We shall follow closely to the work of Hauser and Malhiot[37].

In a flat space with a constant metric, $\eta_{\alpha\beta}$, the general solution for a conformal Killing vector, ξ^{α}, is

$$\xi^{\alpha} = A_{\theta}x^{\theta}x^{\alpha} - \tfrac{1}{2}A^{\alpha}x_{\theta}x^{\theta} + x^{\theta}\omega_{\theta}{}^{\alpha} + \phi_0 x^{\alpha} + B^{\alpha}, \tag{E.1}$$

where $\xi^{(\alpha,\beta)} = (A_{\theta}x^{\theta} + \phi_0)\eta^{\alpha\beta}$, and A_{α}, $\omega_{\alpha\beta} = \omega_{[\alpha\beta]}$, ϕ_0 and B_{α} are real constants. The inner-product, $\xi^{\alpha}P_{\alpha}$, of the conformal Killing vector, ξ^{α}, and a null geodesic, P_{α}, is constant along P_{α}. Therefore, from eq.(E.1), the terms

$$P_{\alpha},\ L_{\alpha\beta} = x_{[\alpha}P_{\beta]},\ \phi = x_{\alpha}P^{\alpha},\ C_{\alpha} = \phi x_{\alpha} - \tfrac{1}{2}P_{\alpha}x_{\theta}x^{\theta}, \tag{E.2}$$

are constant along P_{α}, and so the second order tensors

$$P_{\alpha}P_{\beta},\ L_{\alpha\beta}L_{\gamma\delta},\ \phi^2,\ C_{\alpha}C_{\beta},\ L_{\alpha\beta}P_{\gamma},\ \phi P_{\alpha},\ C_{\alpha}P_{\beta},$$
$$\phi L_{\alpha\beta},\ L_{\alpha\beta}C_{\gamma},\ \phi C_{\alpha}, \tag{E.3}$$

are also constant along P_{α}. Since these are quadratic in the momenta, P_{α}, we see from Lemma K.1 that they define C.K.T.'s. Each term in eq.(E.3) may therefore be written as $Q_A{}^{\alpha\beta}P_{\alpha}P_{\beta}$, where the index A labels the different C.K.T.'s. We shall eventually show that N_n of the expressions in eq.(E.3) are linearly independent, where $N_n = (n-1)(n+2)(n+3)(n+4)/12$.

The equations

$$\eta^{\alpha\beta}P_{\alpha}P_{\beta} = 0,\quad L_{[\alpha\beta}P_{\gamma]} = 0,\quad L_{\alpha[\beta}L_{\gamma\delta]} = 0,$$
$$L_{[\alpha\beta}C_{\gamma]} = 0,\quad \eta^{\alpha\beta}C_{\alpha}C_{\beta} = 0,\quad \phi^2 = 2\eta^{\alpha\beta}C_{\alpha}P_{\beta} + \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\delta}L_{\alpha\gamma}L_{\beta\delta},$$

$$\phi L_{\alpha\beta} = C_{[\alpha}P_{\beta]}\,, \quad 2C_{(\alpha}P_{\beta)} = \eta^{\gamma\delta}L_{\alpha\gamma}L_{\delta\beta}\,,$$

$$\phi P_{\alpha} = \eta^{\beta\gamma}L_{\alpha\beta}P_{\gamma}\,, \quad \phi C_{\alpha} = \eta^{\beta\gamma}L_{\alpha\beta}C_{\gamma}\,, \tag{E.4}$$

split eq.(E.3) into the five sets $\{P_\alpha P_\beta\}$, $\{L_{\alpha\beta}P_\gamma\}$, $\{L_{\alpha\beta}L_{\gamma\delta},\ C_{[\alpha}P_{\beta]}\}$, $\{L_{\alpha\beta}C_\gamma\}$, $\{C_\alpha C_\beta\}$, formed from homogeneous polynomials in x^α of order 0,1,2,3,4 respectively. As these sets are directly analagous to the sets $\{Q_{\alpha\beta}\}$ $\{L_{\alpha\beta\gamma}\}$, $\{M_{\alpha\beta\gamma\delta},\ H_{\alpha\beta}\}$, $\{H_{\alpha\beta\gamma}\}$, $\{I_{\alpha\beta}\}$, which were mentioned in Theorem K.1, there are at most N_n linearly independent members in eq.(E.3).

The only relationship among the members of the set $\{P_\alpha P_\beta\}$ is $\eta^{\alpha\beta}P_\alpha P_\beta = 0$ - which is given in eq.(E.4).

To find the linear dependencies among the members of the set $\{L_{\alpha\beta}P_\gamma\}$ we shall consider the equation $A_{\alpha\beta\gamma}x^\alpha P^\beta P^\gamma = 0$, where $A_{\alpha\beta\gamma} = A_{[\alpha\beta]\gamma}$ is a constant tensor. Then $A_{\alpha\beta\gamma}$ satisfies the equation

$$A_{[\alpha\beta]\gamma} = D_{\alpha[\beta\gamma]} + B_\alpha \eta_{\beta\gamma}\,, \tag{E.5}$$

where $D_{\alpha\beta\gamma}$ and B_α are also constant. The trace of eq.(E.5) gives

$$B_\alpha = D_{\beta\alpha}{}^{\beta}\,. \tag{E.6}$$

Symmetrizing eq.(E.5) over α and β, and taking the trace over β and γ gives

$$(n+1)\,B_\alpha + D_{\beta\alpha}{}^{\beta} = 0\,. \tag{E.7}$$

From eqs.(E.6,7), B_α is zero, whence eq.(E.5) gives $A_{\alpha\beta\gamma}$ as anti-symmetric in all of its indices. This restriction is given in eq.(E.4).

The equation to be considered for the set $\{L_{\alpha\beta}L_{\gamma\delta},\ C_{[\alpha}P_{\beta]}\}$ is

$$A_{\alpha\beta\gamma\delta}x^{\alpha}p^{\beta}x^{\gamma}p^{\delta} + B_{\alpha\beta}C^{\alpha}p^{\beta} = 0 \tag{E.8}$$

where the constant tensors $A_{\alpha\beta\gamma\delta}$ and $B_{\alpha\beta}$ satisfy

$$A_{\alpha\beta\gamma\delta} = A_{[\alpha\beta][\gamma\delta]} = A_{[\gamma\delta][\alpha\beta]}\ ,\quad B_{\alpha\beta} = B_{[\alpha\beta]}\ . \tag{E.9}$$

We shall differentiate eq.(E.8) with respect to x^{ρ} and x^{σ}, obtaining

$$2A_{\rho\beta\sigma\delta}p^{\beta}p^{\delta} + 2\eta_{\delta(\sigma}B_{\rho)\beta}p^{\beta}p^{\delta} = 0\ . \tag{E.10}$$

Thus

$$A_{\rho\beta\sigma\delta} + \eta_{\delta(\sigma}B_{\rho)\beta} = E_{\rho\sigma}\eta_{\beta\delta} + D_{\rho\sigma[\beta\delta]}\ , \tag{E.11}$$

for some $E_{\rho\sigma}$, $D_{\rho\sigma[\beta\delta]}$. The trace over β and δ in eq.(E.11) gives

$$A_{\rho\beta\sigma}{}^{\beta} = nE_{\rho\sigma} = nE_{(\rho\sigma)}\ . \tag{E.12}$$

Next, symmetrize eq.(E.11) over β and δ, and take the trace over β and σ,

$$-A_{\rho\beta\delta}{}^{\beta} + \tfrac{1}{2}(n+2)B_{\rho\delta} = 2E_{\rho\delta}\ . \tag{E.13}$$

Thus $B_{\rho\delta}$ is zero, being symmetric and anti-symmetric in ρ and δ. From eq.(E.12) and eq.(E.13), $E_{\rho\sigma}$ is zero, and so from eq.(E.10), $A_{\alpha\beta\gamma\delta}$ is completely antisymmetric in β, γ and δ.

Since $\eta^{\rho\sigma}\partial_{\rho}\partial_{\sigma}\partial_{\mu}(A_{\alpha\beta\gamma}L^{\alpha\beta}C^{\gamma}) = (2A_{\alpha\beta\gamma}\eta^{\alpha\gamma}\eta_{\mu\sigma} + 2nA_{\beta\mu\sigma})p^{\beta}p^{\sigma}$, the next equation to consider is

$$A_{\alpha\beta\gamma}\eta^{\alpha\gamma}\eta_{\mu\sigma} + nA_{\beta\mu\sigma} = B_{\mu}\eta_{\beta\sigma} + D_{\mu[\beta\sigma]}\ . \tag{E.14}$$

The trace of eq.(E.14) over β and σ gives

$$nB_{\mu} = (n+1)A_{\beta\mu}{}^{\beta}\ . \tag{E.15}$$

By symmetrizing over β and σ in eq.(E.14), and taking the trace over β and μ, we find that

$$2B_{\sigma} = A_{\beta\sigma}{}^{\beta} . \tag{E.16}$$

Thus $B_{\sigma} = 0 = A_{\beta\sigma}{}^{\beta}$. Then from eq.(E.14), $A_{\alpha\beta\gamma}$ is anti-symmetric in all of its indices.

The final equation to consider is $A_{\alpha\beta}C^{\alpha}C^{\beta} = 0$, where $A_{\alpha\beta} = A_{(\alpha\beta)}$. We see that

$$\eta^{\rho\sigma}\eta^{\mu\nu}\partial_{\rho}\partial_{\sigma}\partial_{\mu}\partial_{\nu}(A_{\alpha\beta}C^{\alpha}C^{\beta}) = n(n-2)A_{\alpha\beta}P^{\alpha}P^{\beta} ,$$

and so for $n \geqslant 3$ we have $A_{\alpha\beta} = \lambda\eta_{\alpha\beta}$ for some constant λ. This finally shows that the only linear relationships between the C.K.T.'s listed in eq.(E.3) are given in eq.(E.4).

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